Partial Data Extraction via Noisy Histogram Query: The Information Theoretic Bounds

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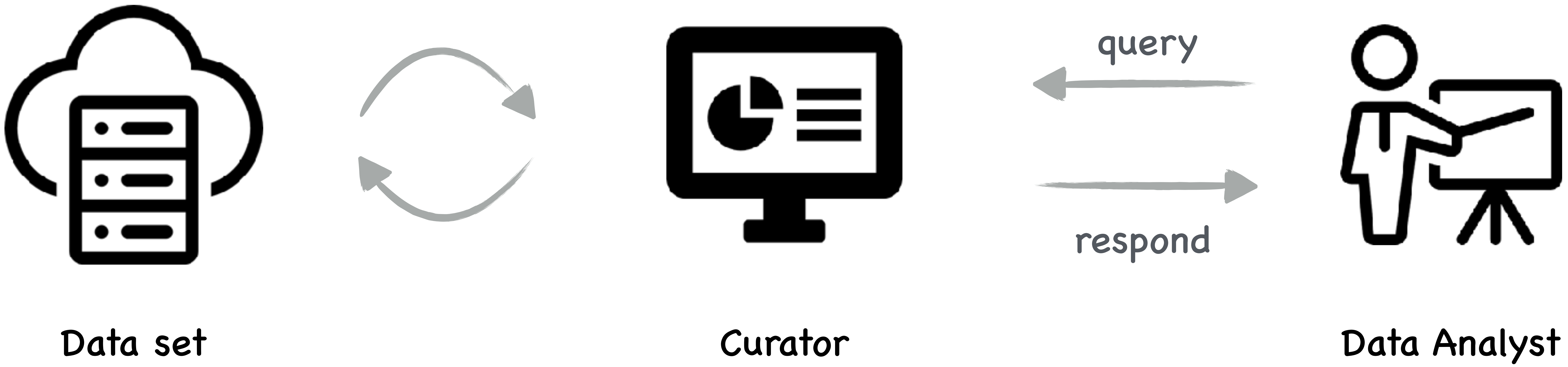
National Taiwan University

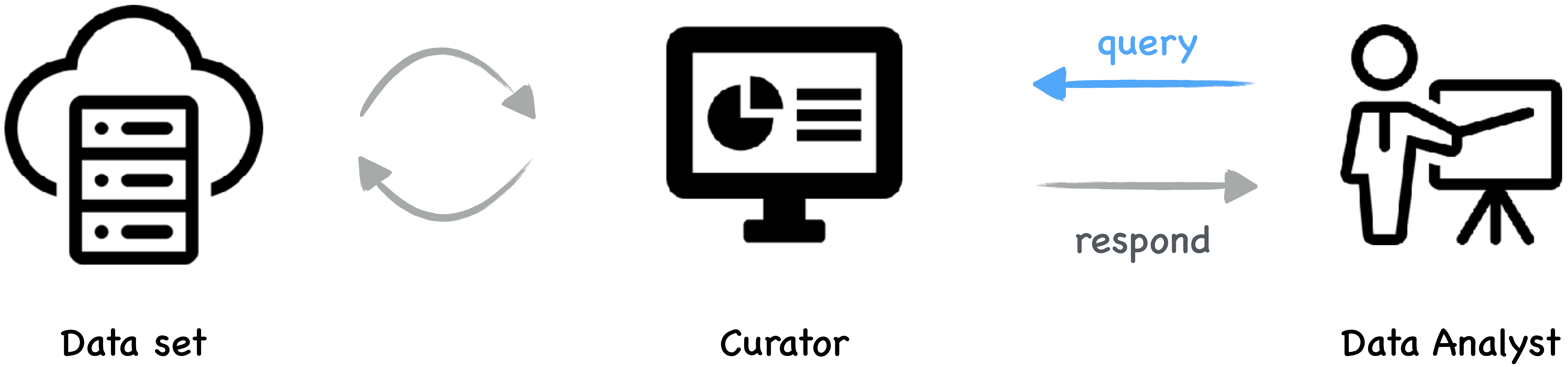
Jun, 2017

Query Model



- Query with the curator

Query Model



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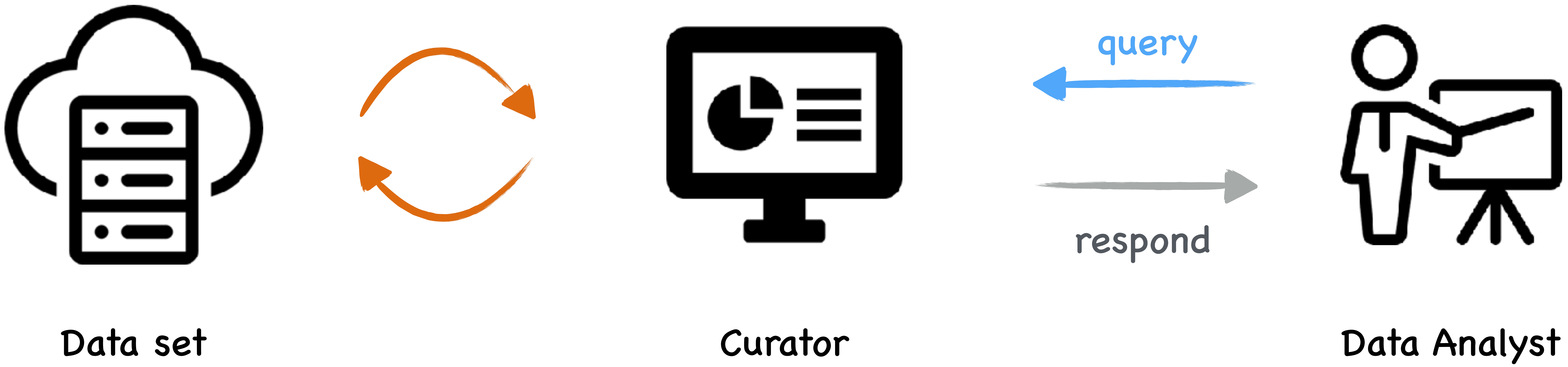

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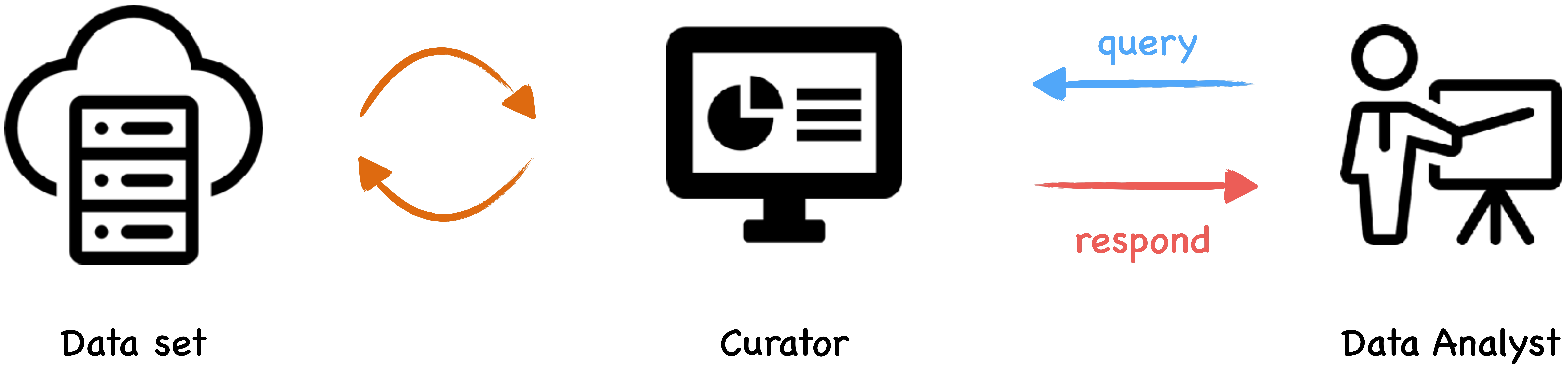

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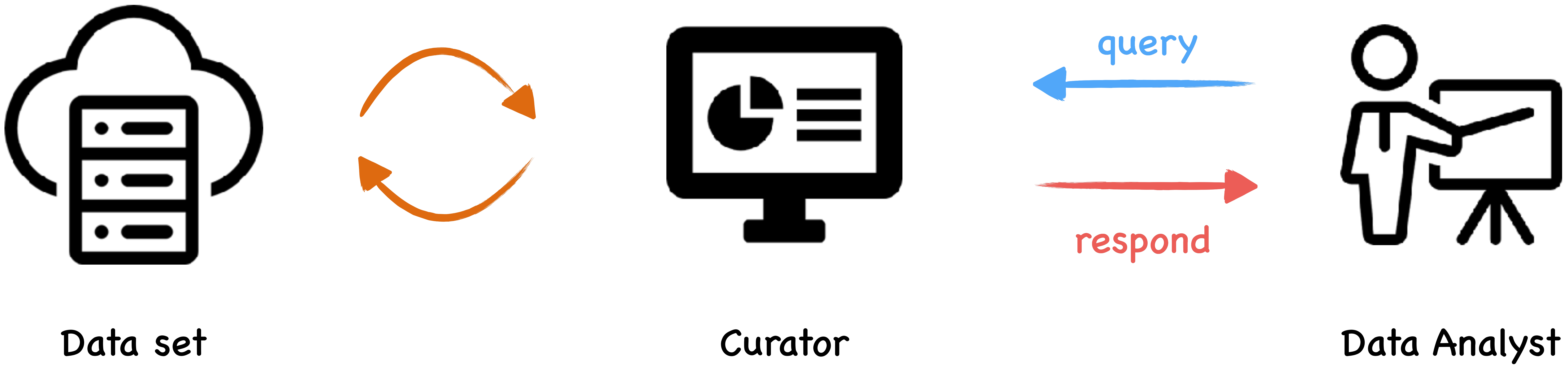
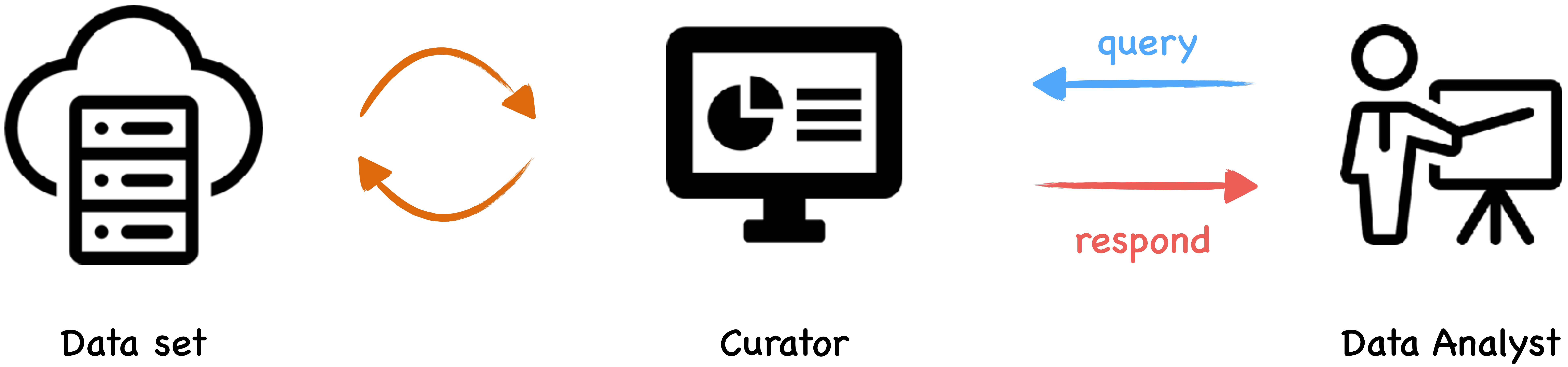


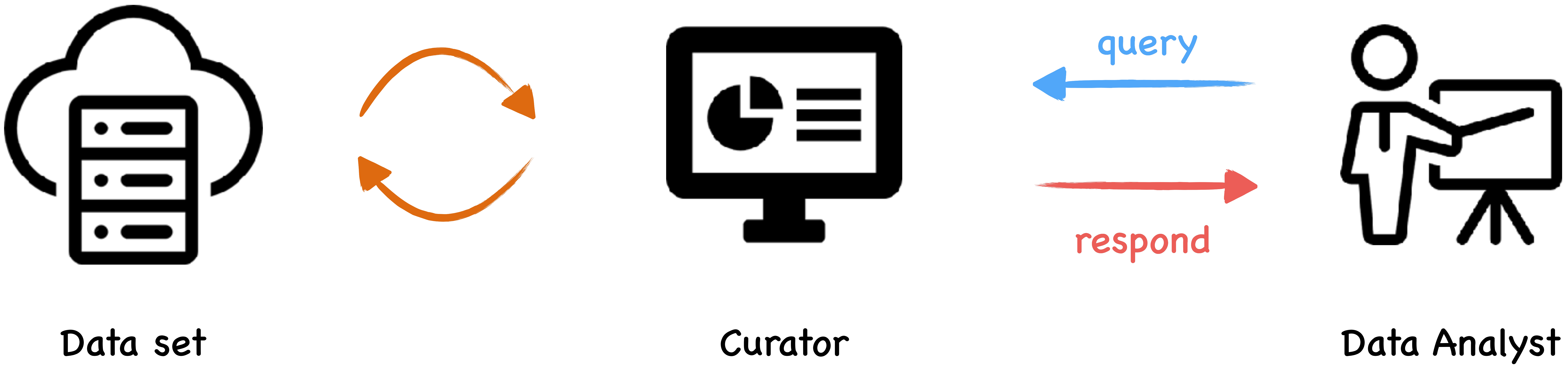
- Query with the curator
- Certain types of queries are allowed

Query Model



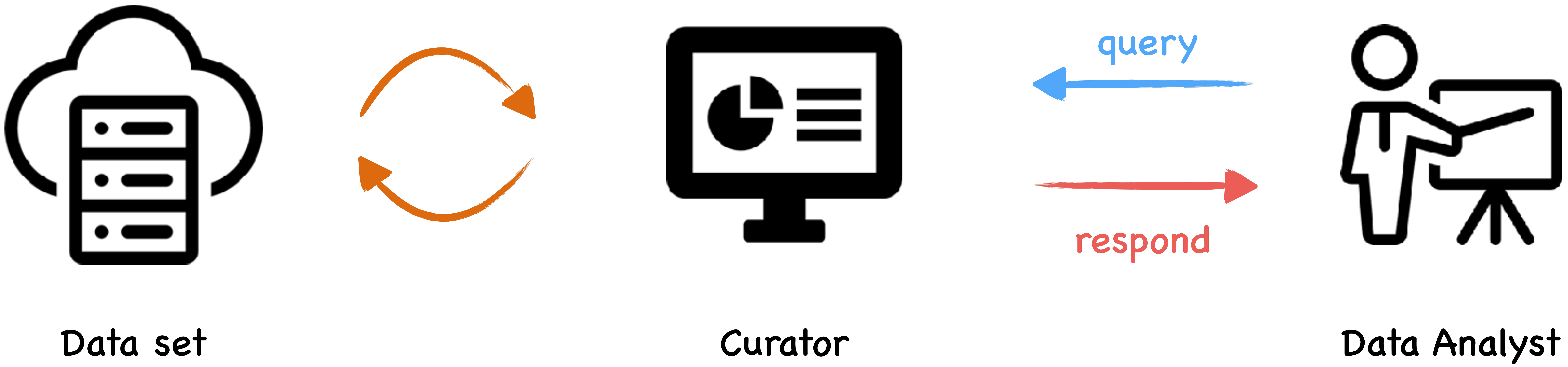
- Query with the curator
- Certain types of queries are allowed
  - Subset query

Query Model



- Query with the curator
- Certain types of queries are allowed
  - Subset query
  - Statistical information of subset

Query Model



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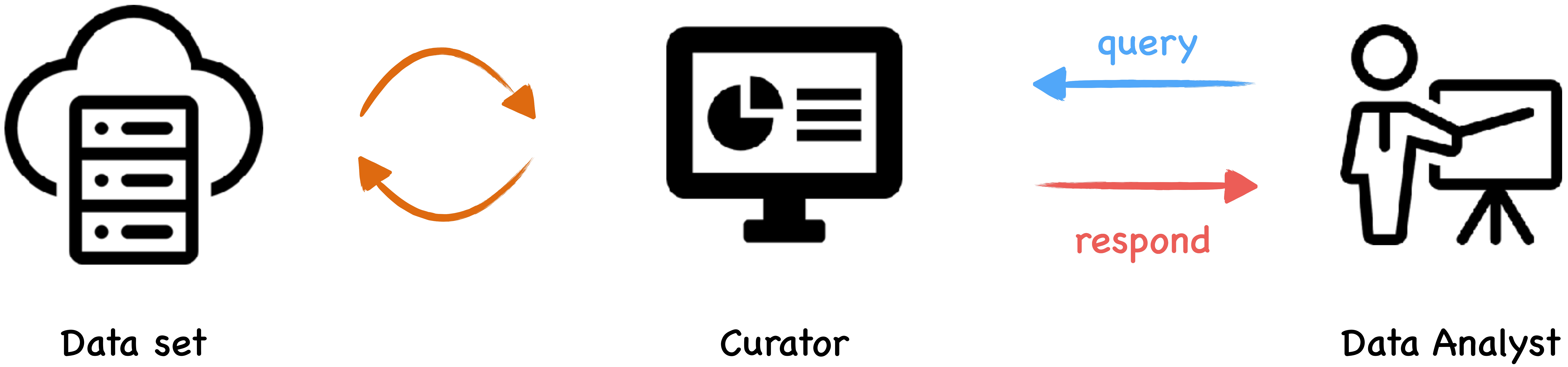
- Example :

Query Model



- Query with the curator
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- Example :

  A. Numerical data : statistical mean, variance etc.

  B. Categorical data : counting number, histogram etc.

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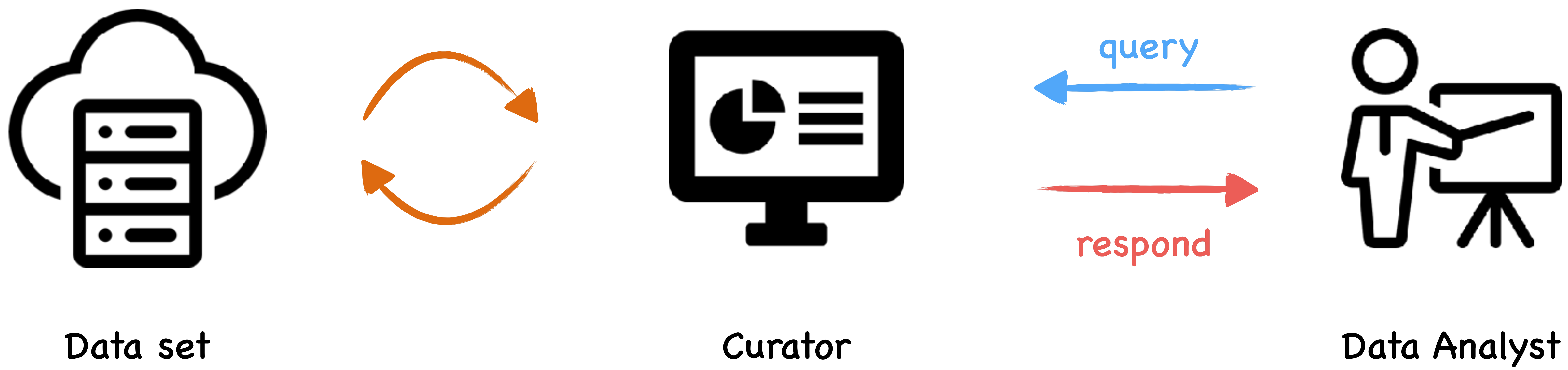

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- Example :

  A. ~~Numerical data : statistical mean, variance etc.~~

  B. Categorical data : counting number, histogram etc.

Histogram Query

- Histogram Query

Users	Blood
1	A
2	A
3	B
4	AB
5	O
6	O

Histogram Query

- Histogram Query

Users	Blood
1	A
2	A
3	B
4	AB
5	O
6	O

User{1,2,3,4}

- Histogram Query

Users	Blood
1	A
2	A
3	B
4	AB
5	O
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User{1,2,3,4}

- Histogram Query

Users	Blood
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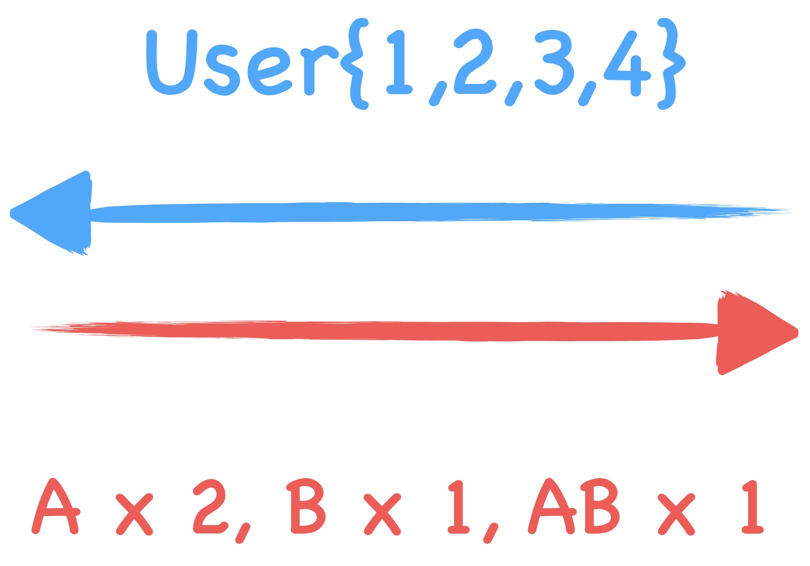

Histogram Query

- Histogram Query

Users	Blood
1	A
2	A
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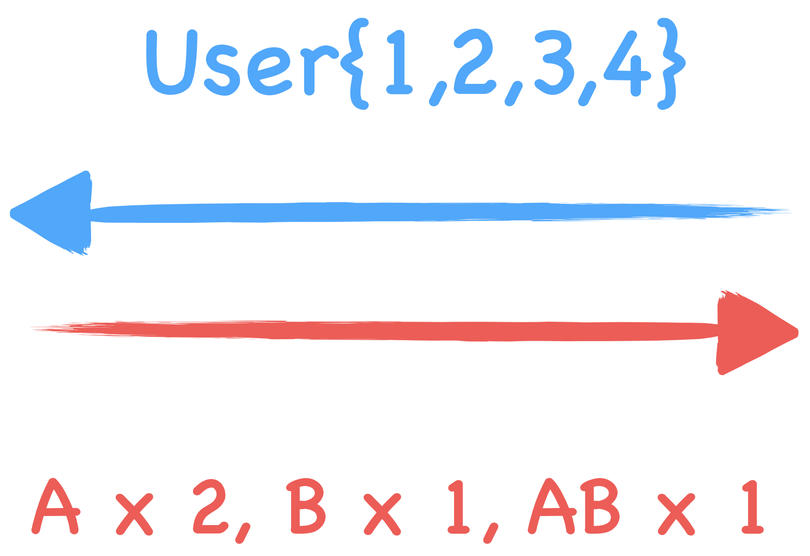

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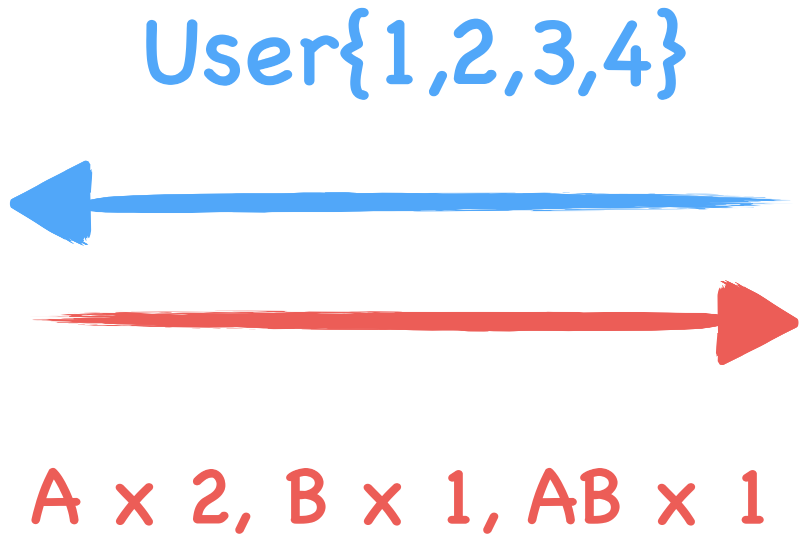

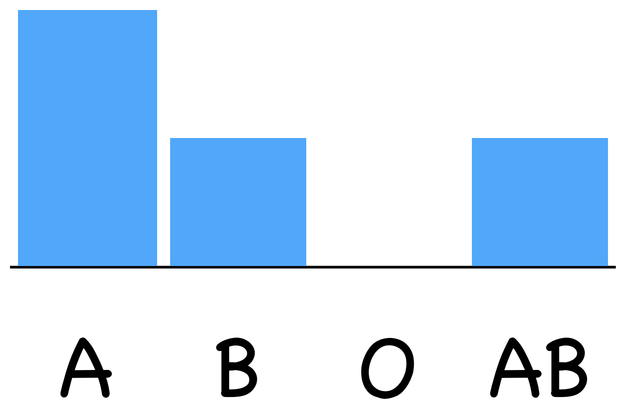

The Noisy Response

- Histogram Query

Users	Blood
1	A
2	A
3	B
⋮	⋮
n	O

The Noisy Response

- Histogram Query

Users	Blood
1	A
2	A
3	B
⋮	⋮
n	O

The Noisy Response

- Histogram Query

Users	Blood
1	A
2	A
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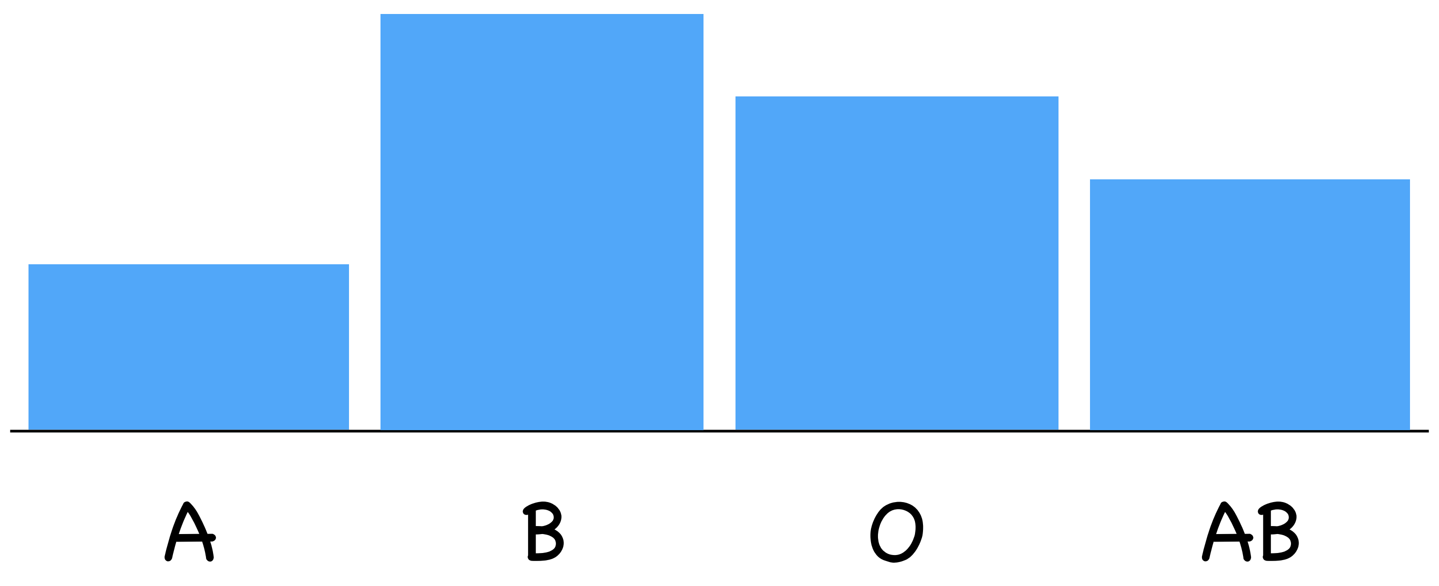
Honest Response

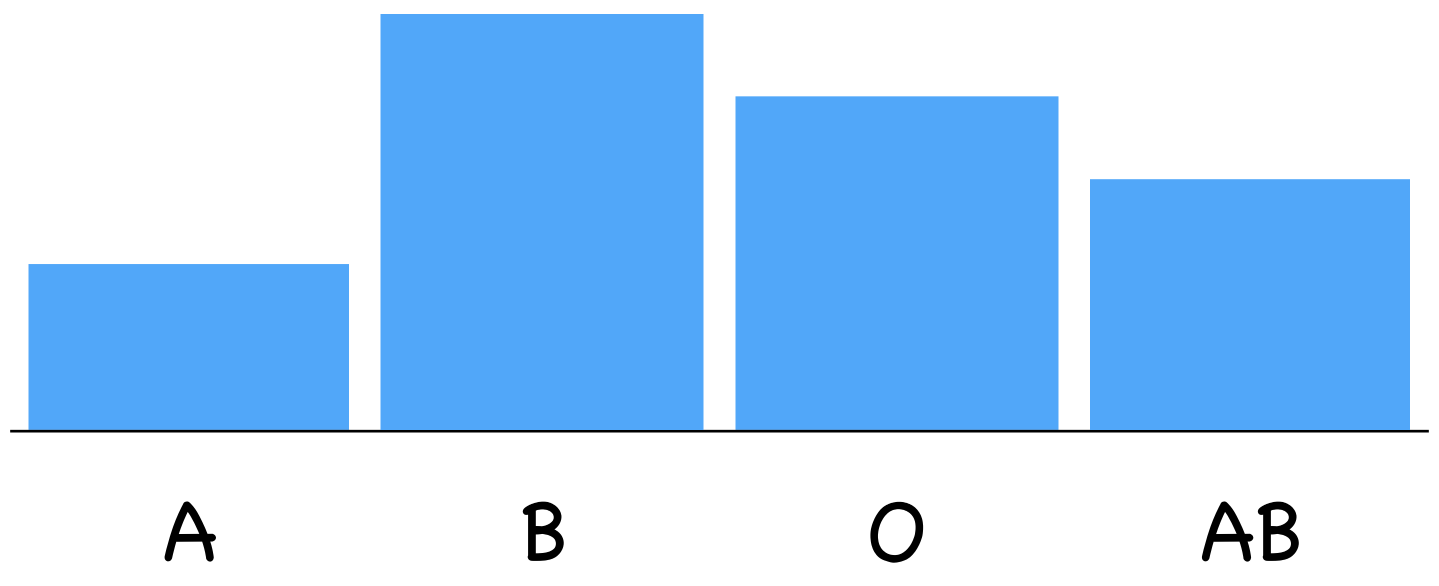


The Noisy Response

- Histogram Query
- Noisy response : ex. to guarantee stronger privacy

Users	Blood
1	A
2	A
3	B
⋮	⋮
n	O

Honest Response



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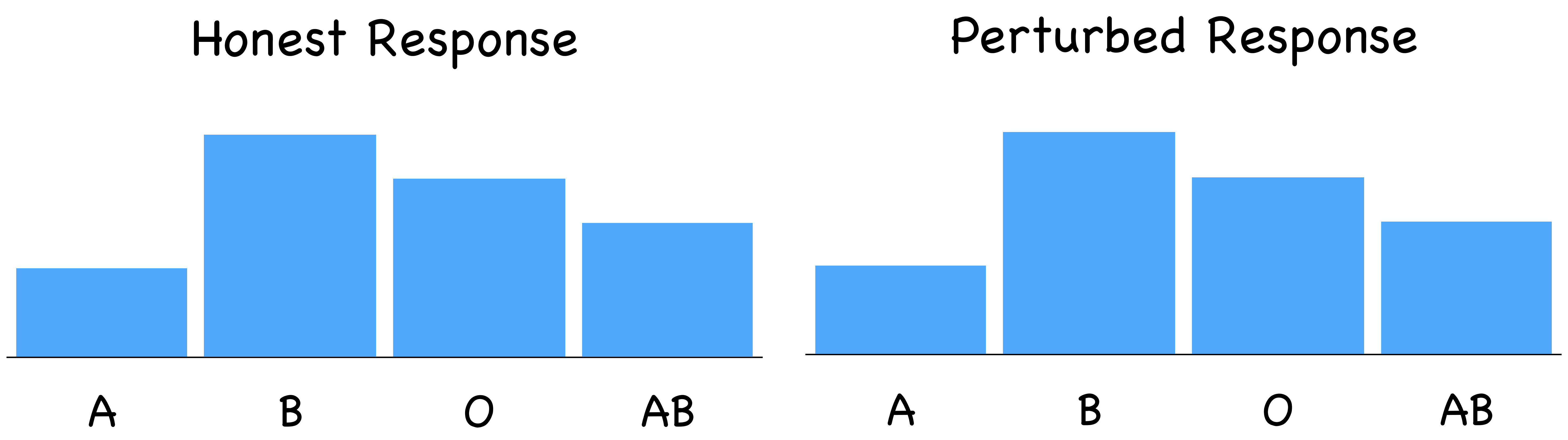

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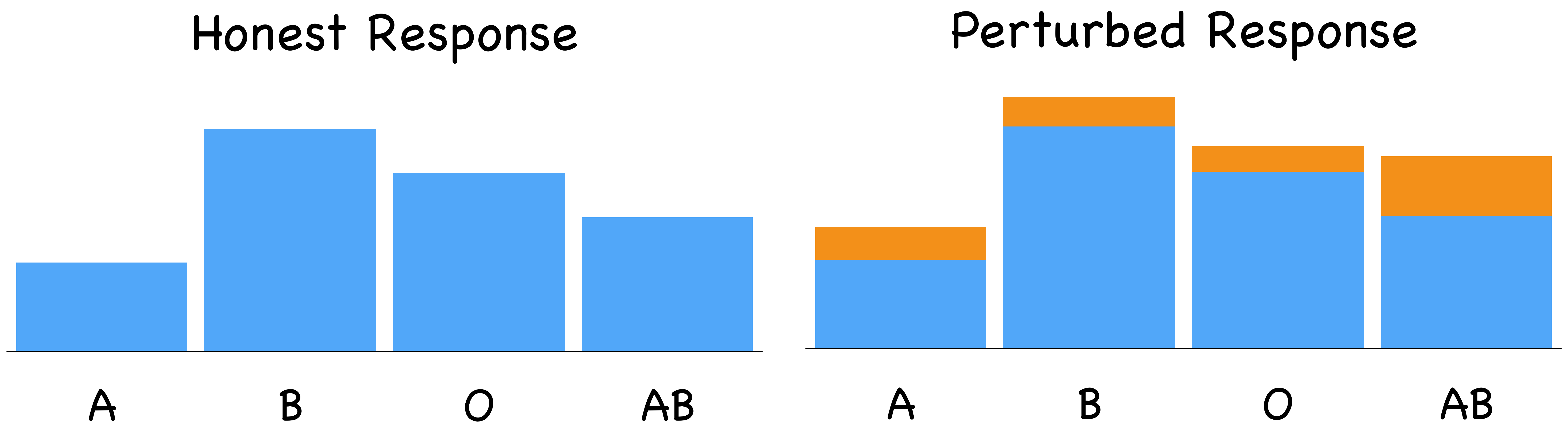

The Noisy Response

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Users	Blood
1	A
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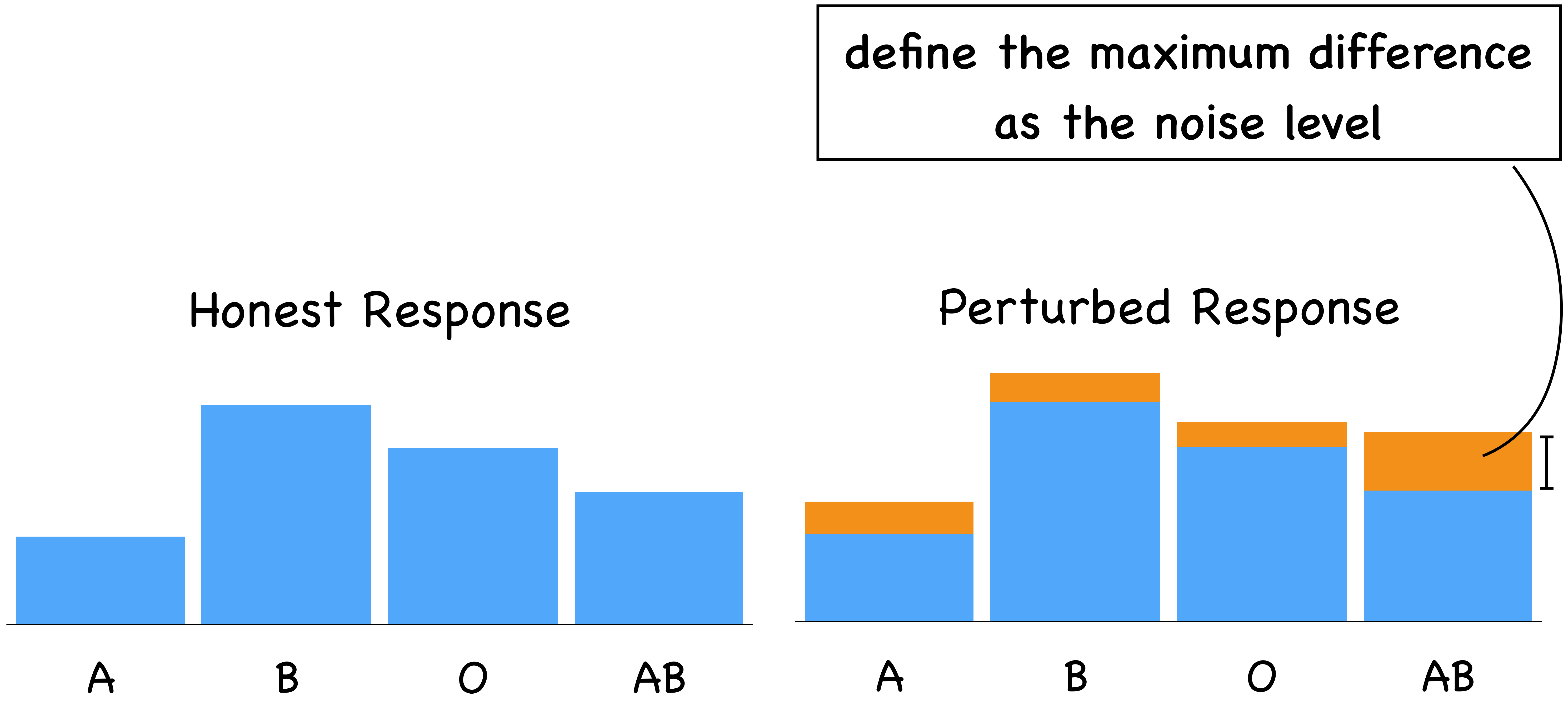
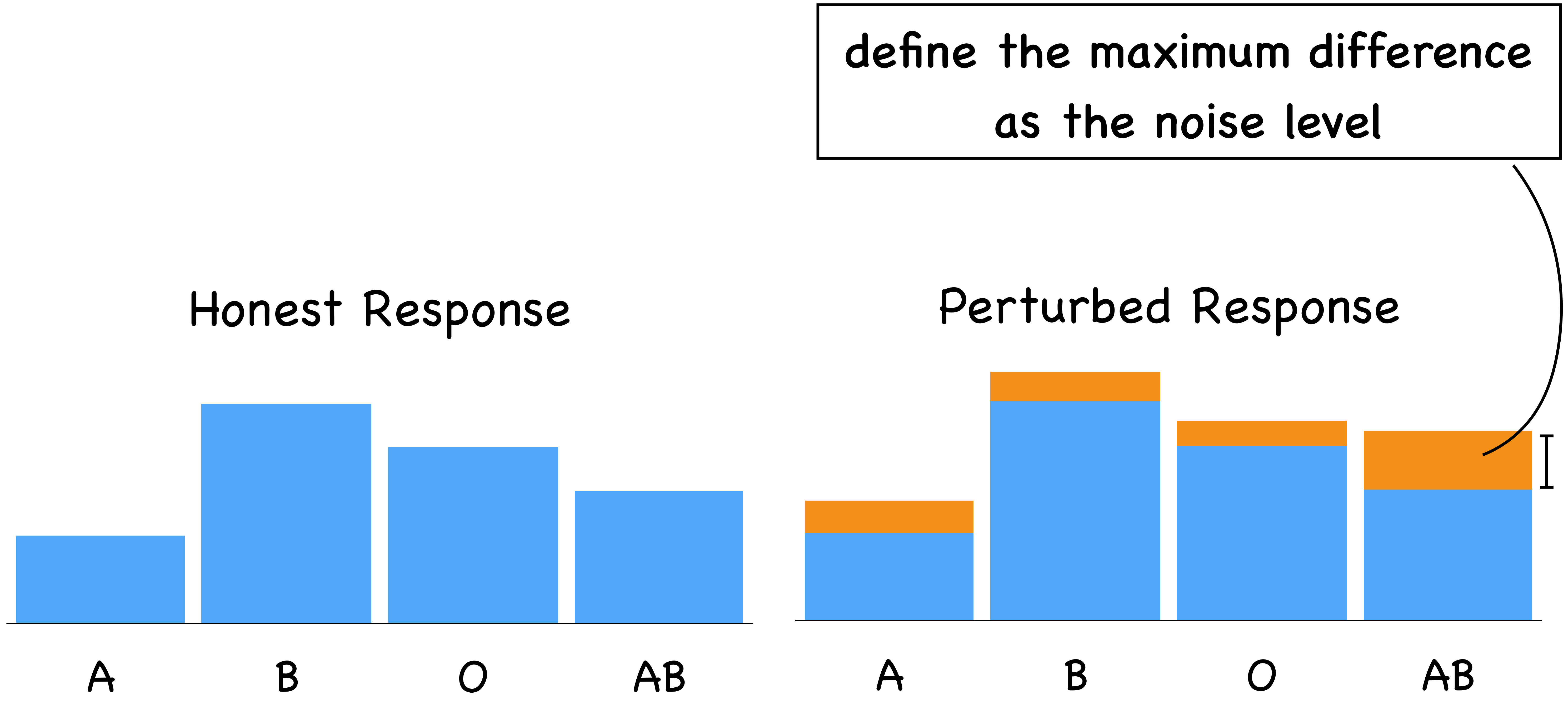


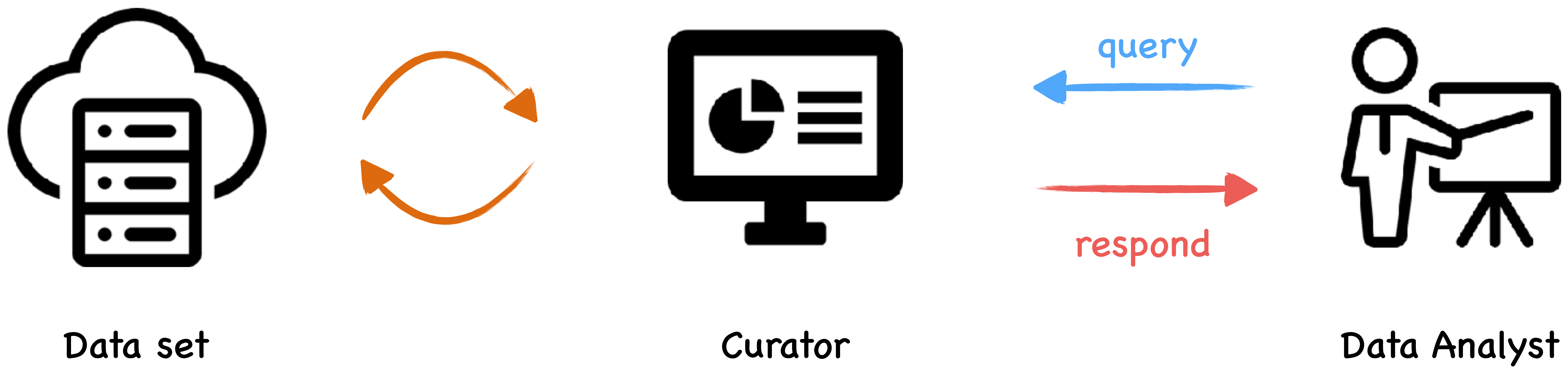
The Noisy Response

- Histogram Query
- Noisy response : ex. to guarantee stronger privacy
- The added noise is at most δ_n

Users	Blood
1	A
2	A
3	B
⋮	⋮
n	O



Problem Statement



- Goal : to extract the data set partially

[1] I.-H. Wang, et. al "Data extraction via histogram and arithmetic mean queries: Fundamental limits and algorithms," Proceedings of IEEE International Symposium on Information Theory 2016

[2] Ahmed El Alaoui , et. al "Decoding from Pooled Data: Phase Transitions of Message Passing ," Proceedings of IEEE International Symposium on Information Theory 2017

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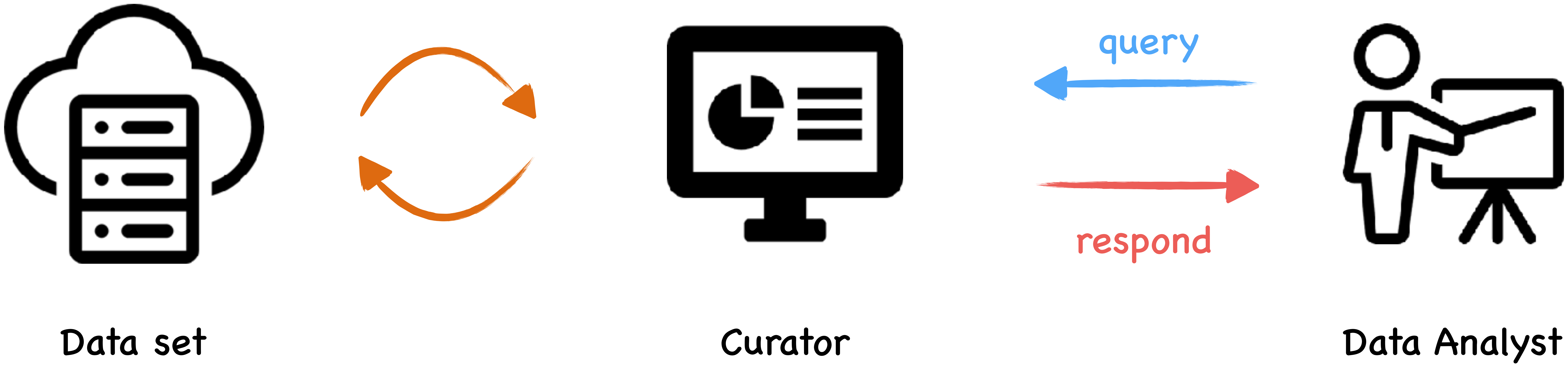

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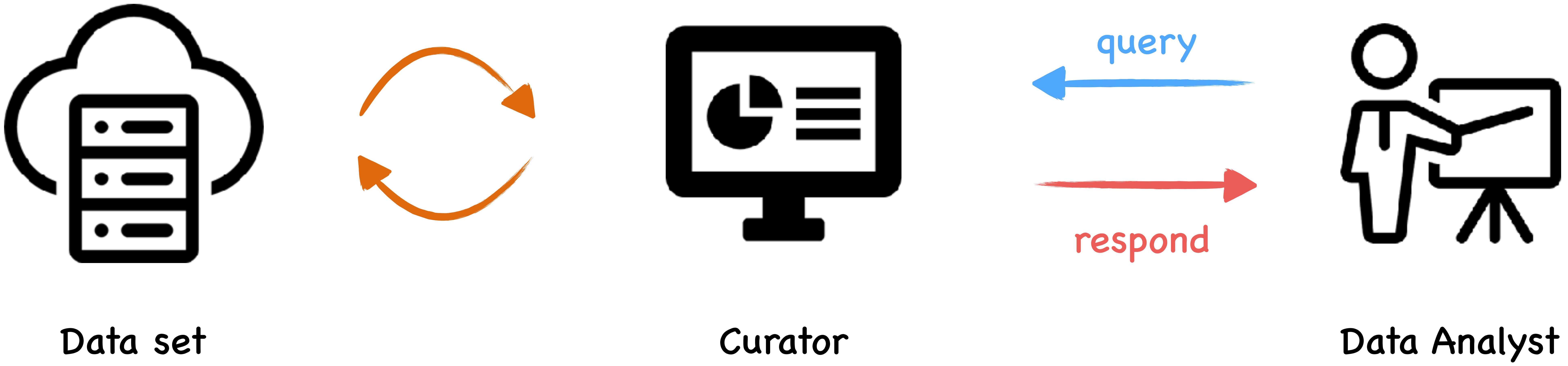

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- Key question : how many queries does the analyst required ?

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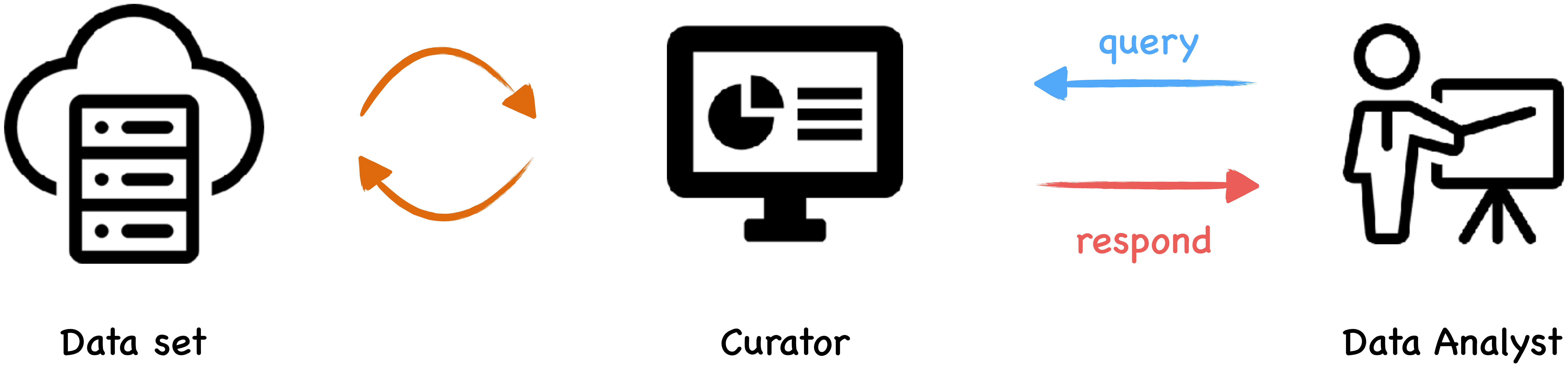

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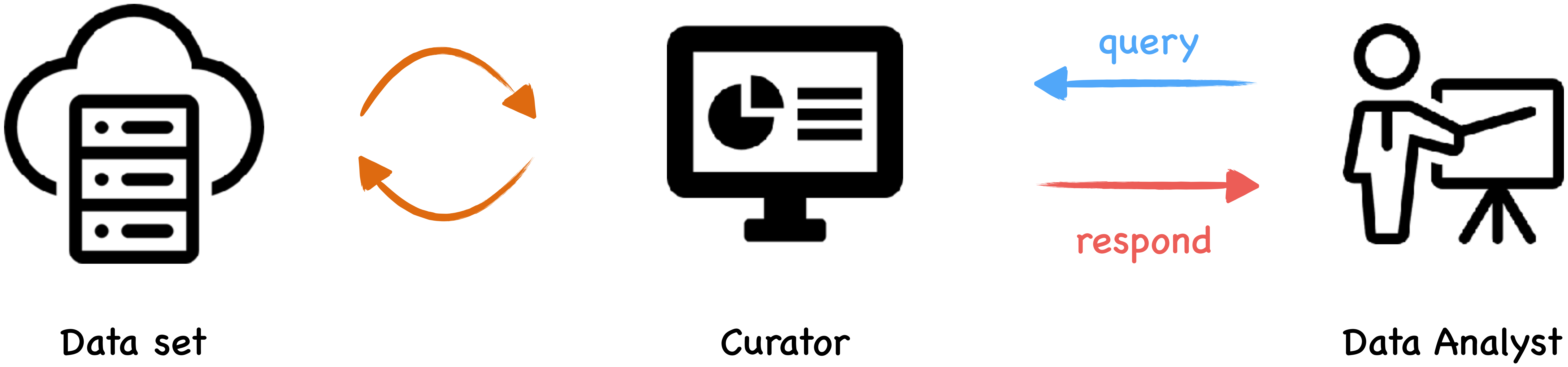

- Goal : to extract the data set partially
  - motivation: privacy, cost of data extraction, etc.
- Key question : how many queries does the analyst required ?
- Query complexity : minimum number of queries to reconstruct the data set
- In noiseless case, i.e. $\delta_n = 0$, the query complexity in [1] is proven to be $\Theta\left(\frac{n}{\log n}\right)$
  Also, in [2], an AMP algorithm is proposed to decode the data set

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Partial Data Reconstruction

- $\mathbf{x}$: original data set

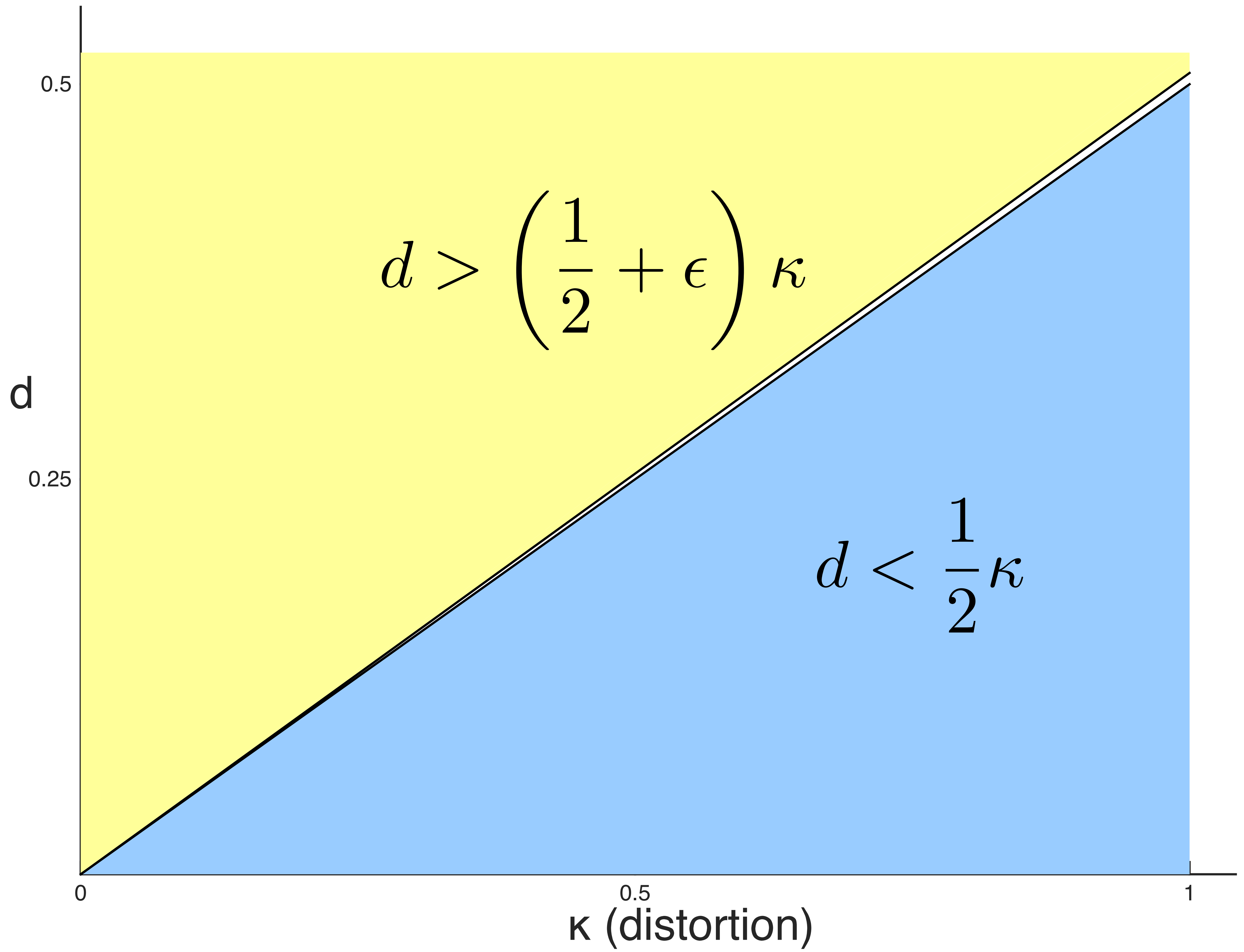
  $\hat{\mathbf{x}}$: recovered data set

- k_n-distortion : $d_{\text{Hamming}}(\mathbf{x}, \hat{\mathbf{x}}) \leq k_n$

$$\mathbf{x} \quad \begin{bmatrix} A \\ B \\ A \\ O \\ AB \\ \vdots \\ O \end{bmatrix} \qquad \hat{\mathbf{x}} \quad \begin{bmatrix} \color{red}{B} \\ B \\ A \\ \color{red}{AB} \\ AB \\ \vdots \\ O \end{bmatrix}$$

Main Result

$$\delta_n = \Theta\left(n^d\right), \; k_n = \Theta\left(n^\kappa\right)$$



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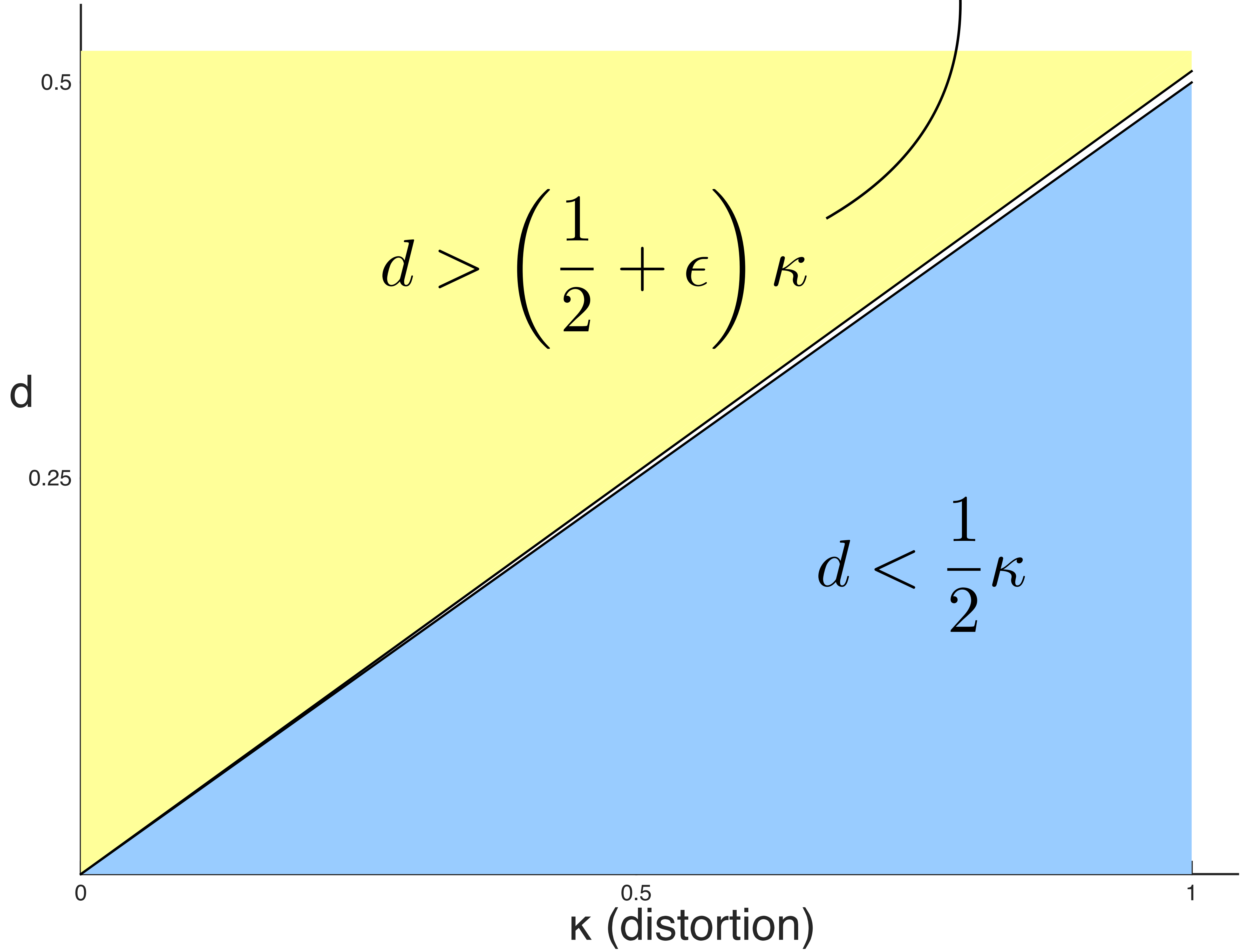
$$\delta_n = \Theta\left(n^d\right),\ k_n = \Theta\left(n^\kappa\right)$$

query complexity: non-polynomial



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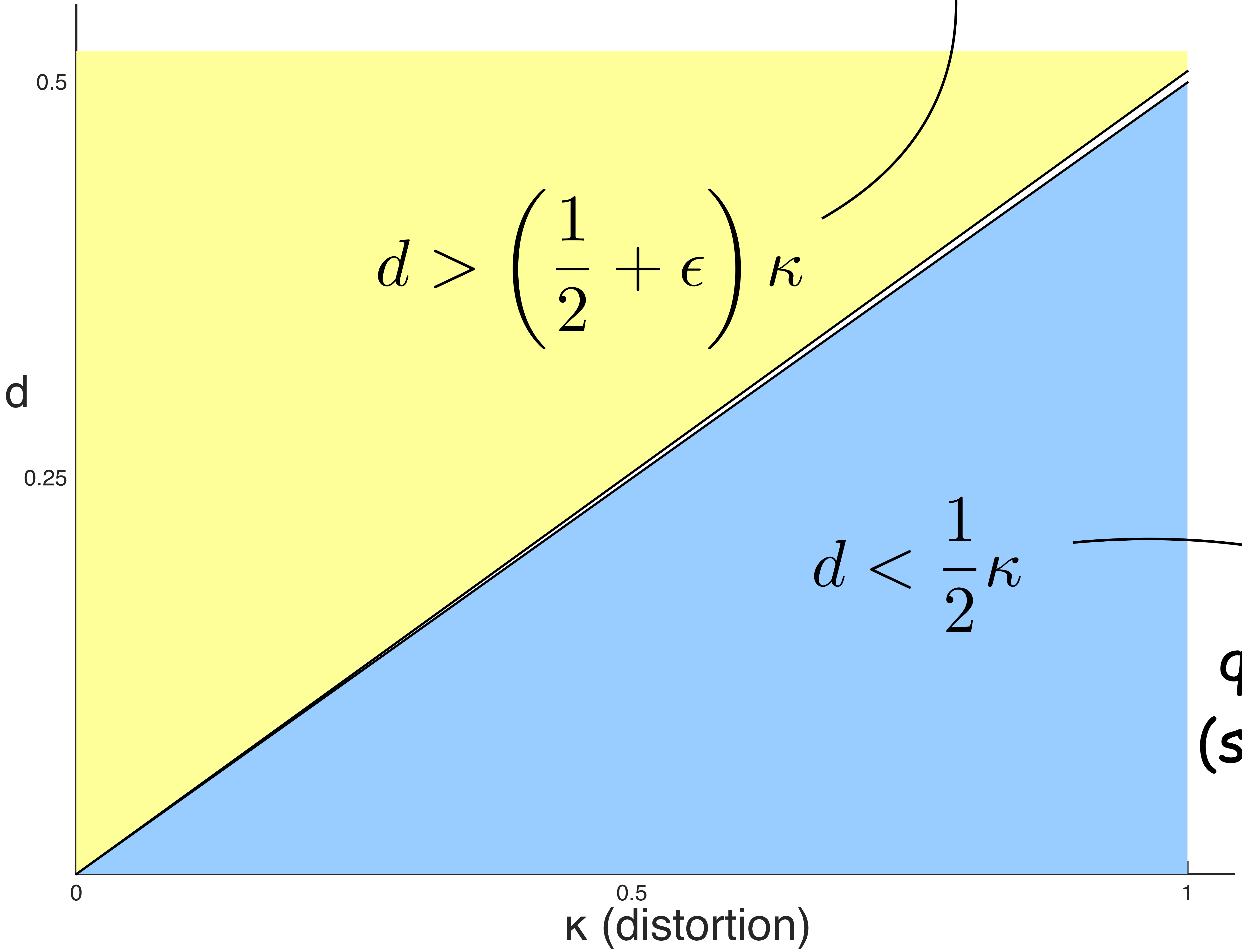
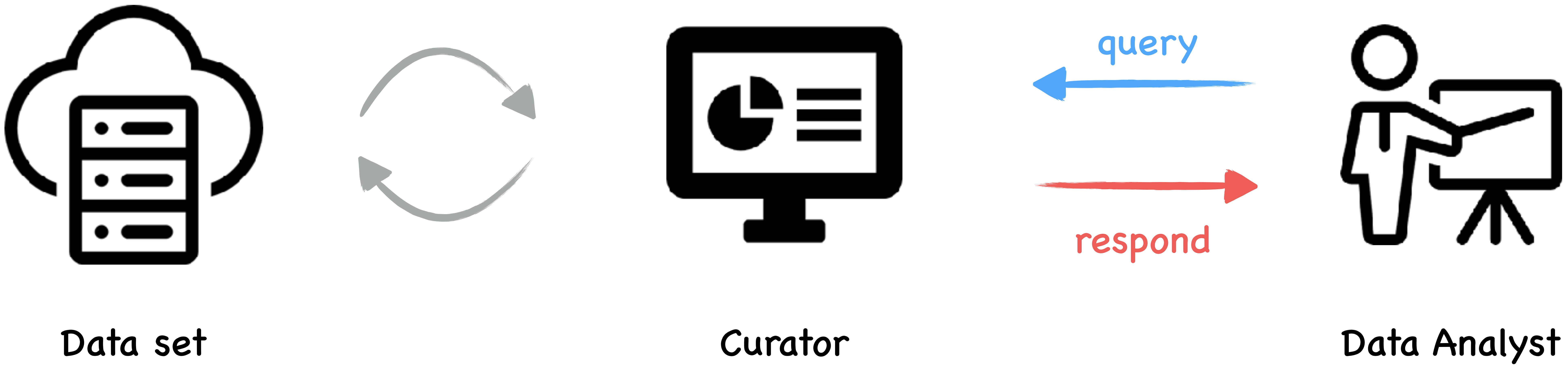


query complexity: sub-linear
(same as the noiseless case !)

The rest of the talk…

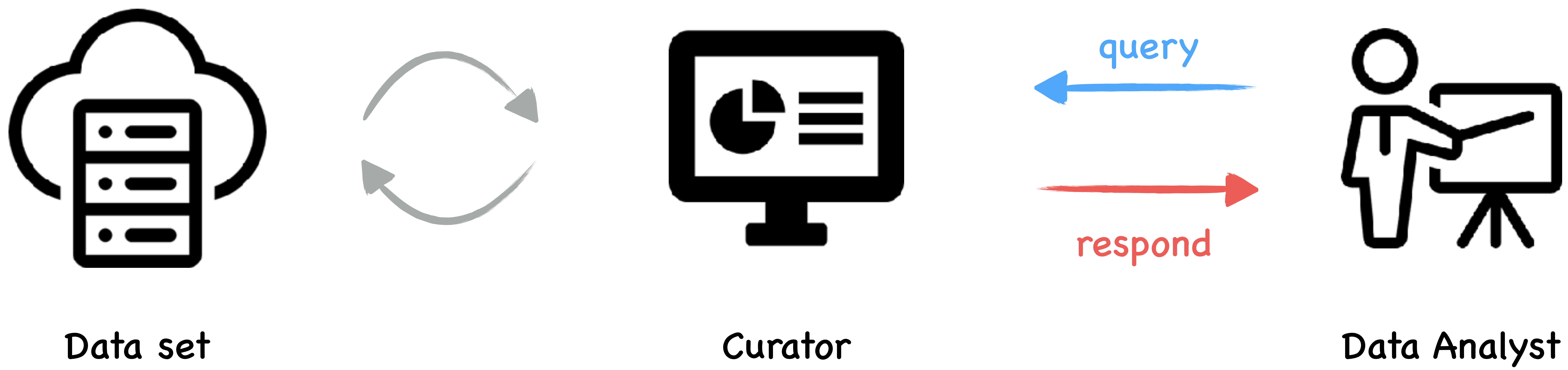
- Problem Formulation
  - Data extraction as a linear inverse problem
- Sketch of Proof :
  A. Regime 1 : Impossibility of Poly-n Query
  B. Regime 2 : The Fundamental Limit of Query Complexity
- Summary

Histogram Query as Linear Multiplication



Users	Blood
1	O
2	B
3	B
⋮	⋮
n	A

Histogram Query as Linear Multiplication

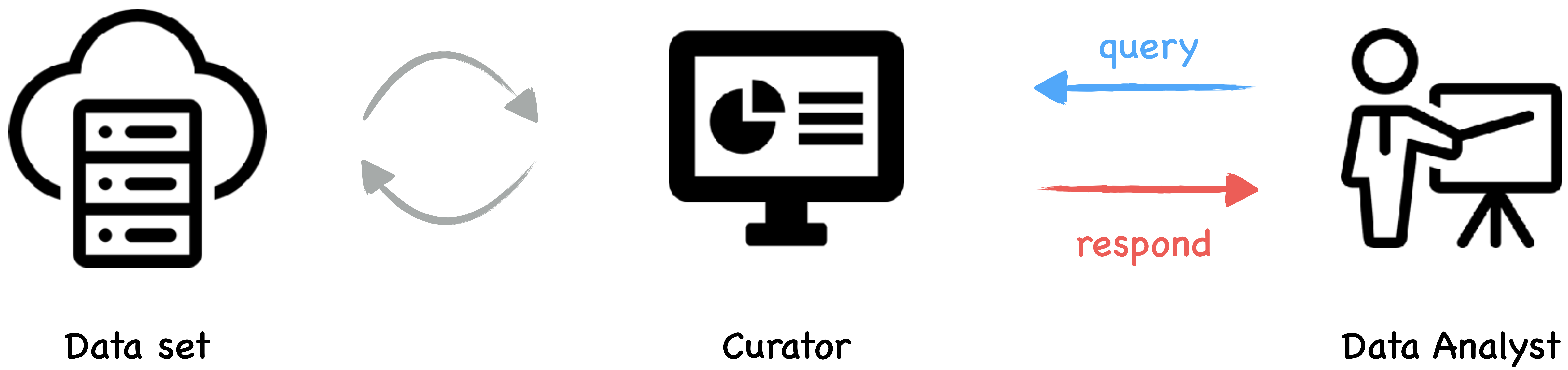


A, B, AB, O

$$n\left\{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \end{bmatrix}\right.$$

$$\mathbf{X}$$

Histogram Query as Linear Multiplication



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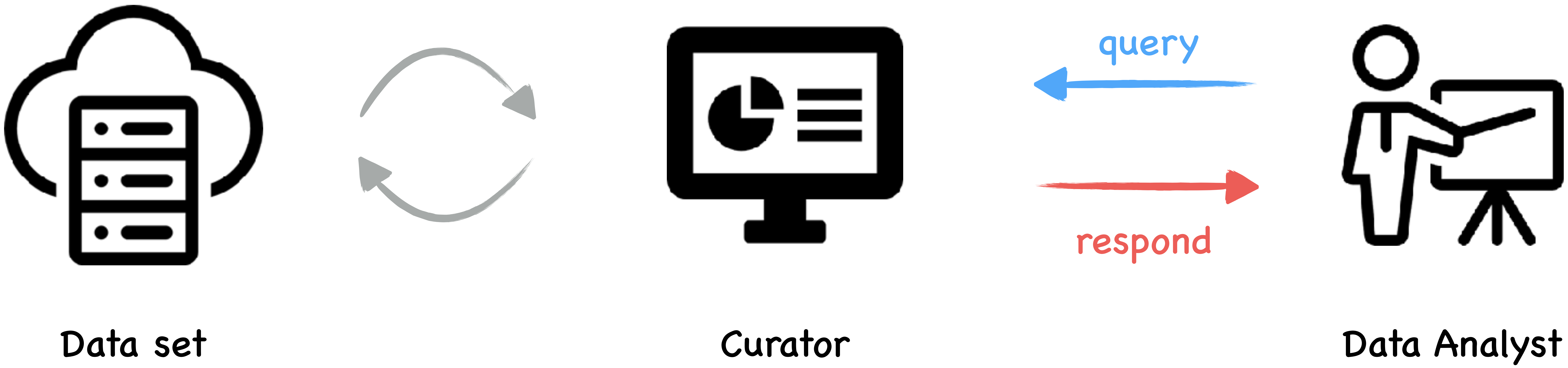
User{1,2,n}

$$\boldsymbol{q}_i^\intercal = \underbrace{\begin{bmatrix}1 & 1 & 0 & \cdots & 0 & 1\end{bmatrix}}_{n}$$

Histogram Query as Linear Multiplication



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$$\boldsymbol{y}_i = \boldsymbol{q}_i^\intercal \mathbf{X}$$

Histogram Query as Linear Multiplication

Data set | Curator | Data Analyst

A, B, AB, O

$$
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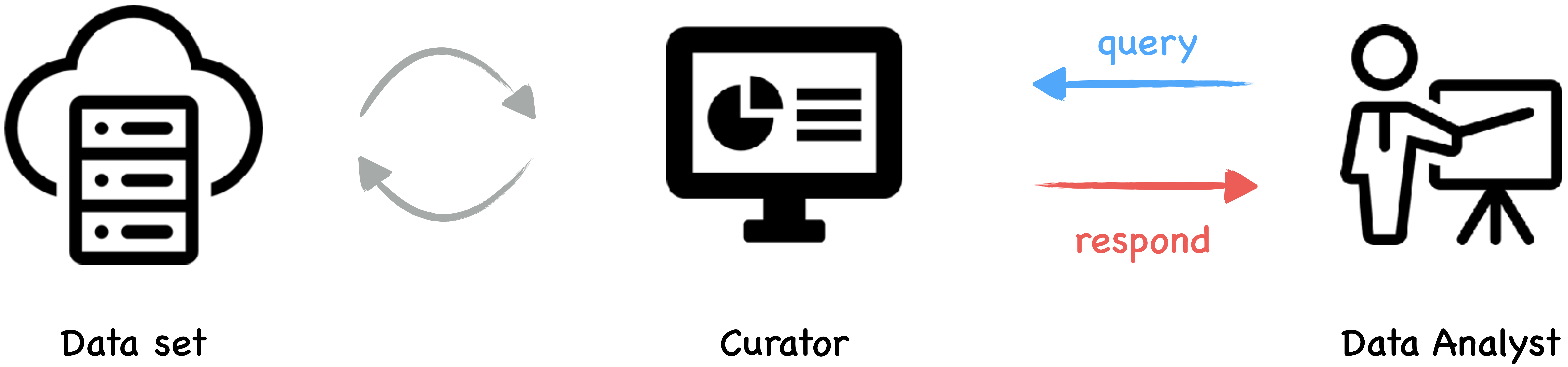
$$\mathbf{X}$$

User{1,2,n}

$$\boldsymbol{q}_i^\intercal = \underbrace{\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 1 \end{bmatrix}}_{n}$$

$$\boldsymbol{y}_i = \boldsymbol{q}_i^\intercal \mathbf{X} + \Delta_i$$

Histogram Query as Linear Multiplication



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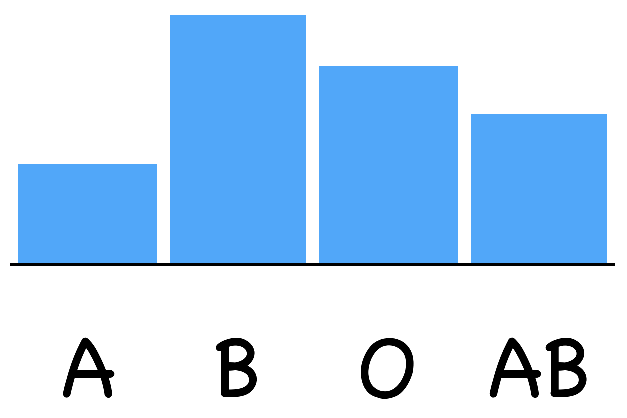
A, B, O, AB

$$\mathbf{y}_i = [10, 20, 18, 28]$$



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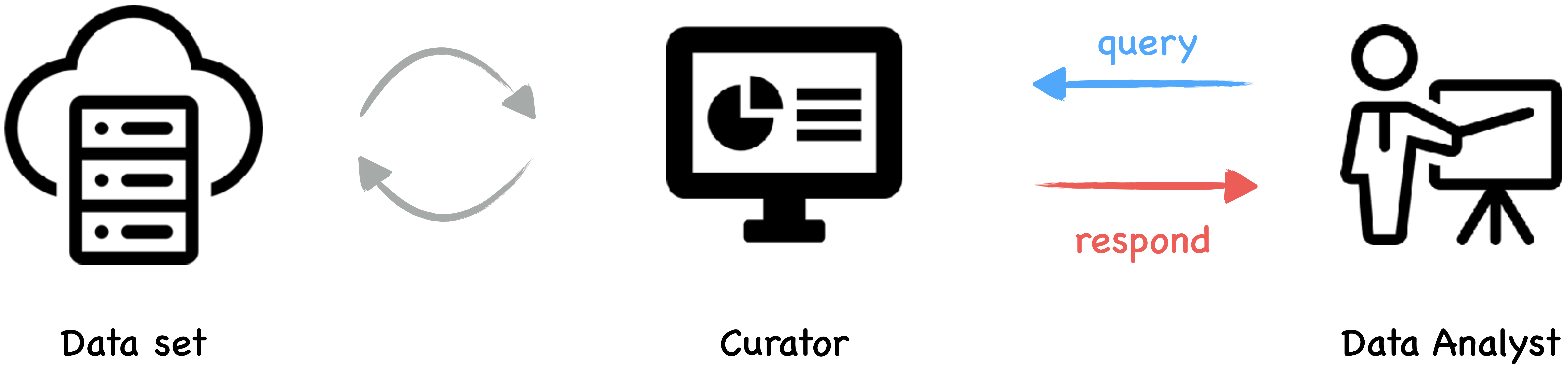

A, B, AB, O

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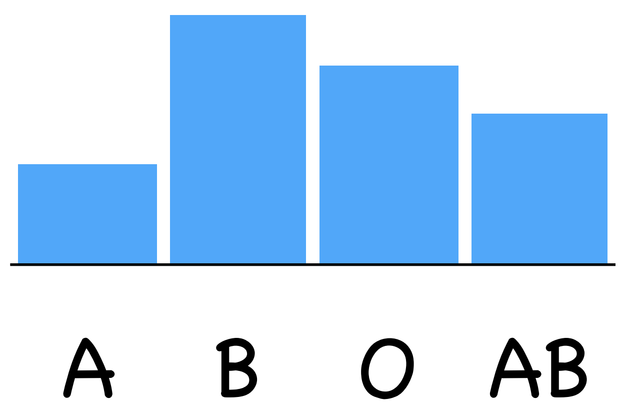

Histogram Query as Linear Multiplication

Data set　　　　Curator　　　　Data Analyst

A, B, AB, O

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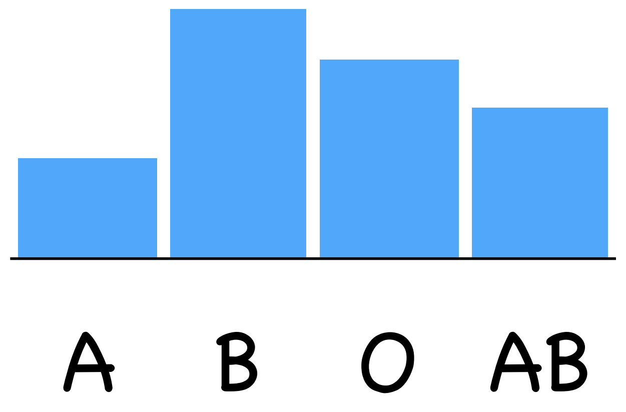

Histogram Query as Linear Multiplication

Data set

Curator

Data Analyst

Decode column by column

A, B, AB, O

$$n \left\{ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \end{bmatrix} \right.$$

$\mathbf{X}$

User{1,2,n}

$$\boldsymbol{q}_i^\intercal = \underbrace{\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 1 \end{bmatrix}}_{n}$$

$$\boldsymbol{y}_i = \boldsymbol{q}_i^\intercal \mathbf{X} + \Delta_i$$

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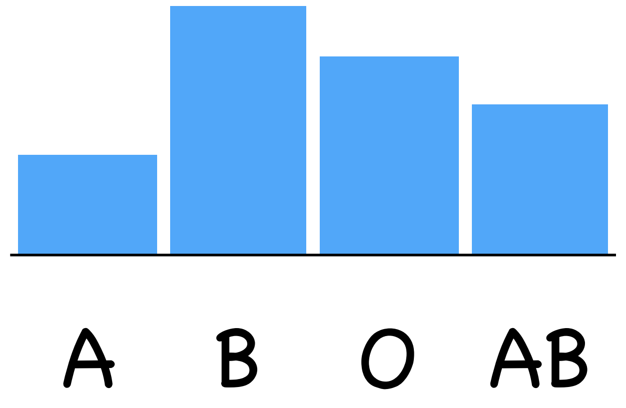
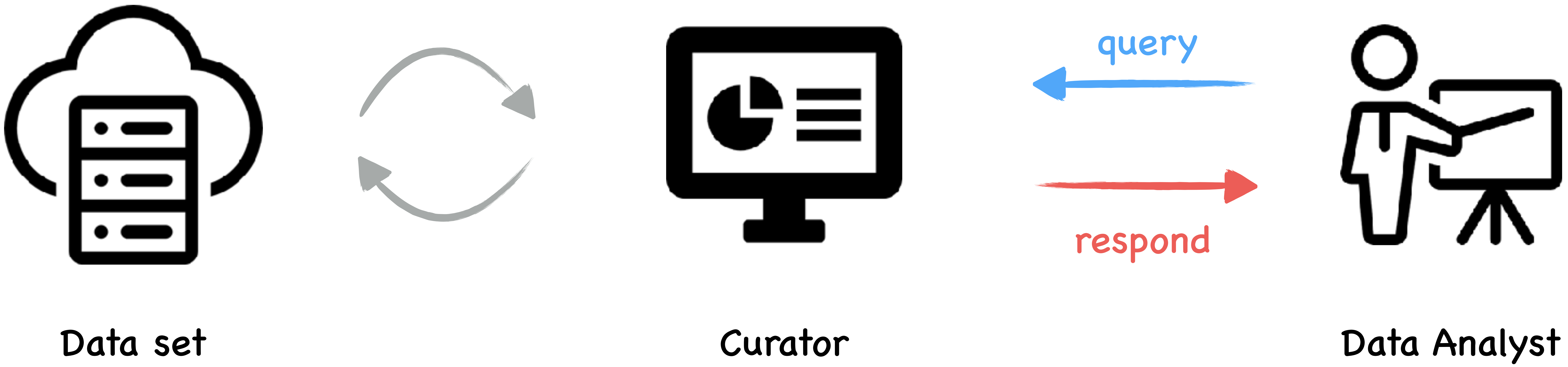


Histogram Query as Linear Multiplication



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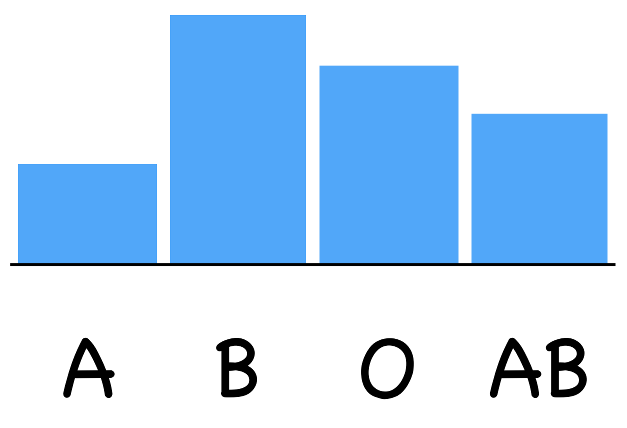

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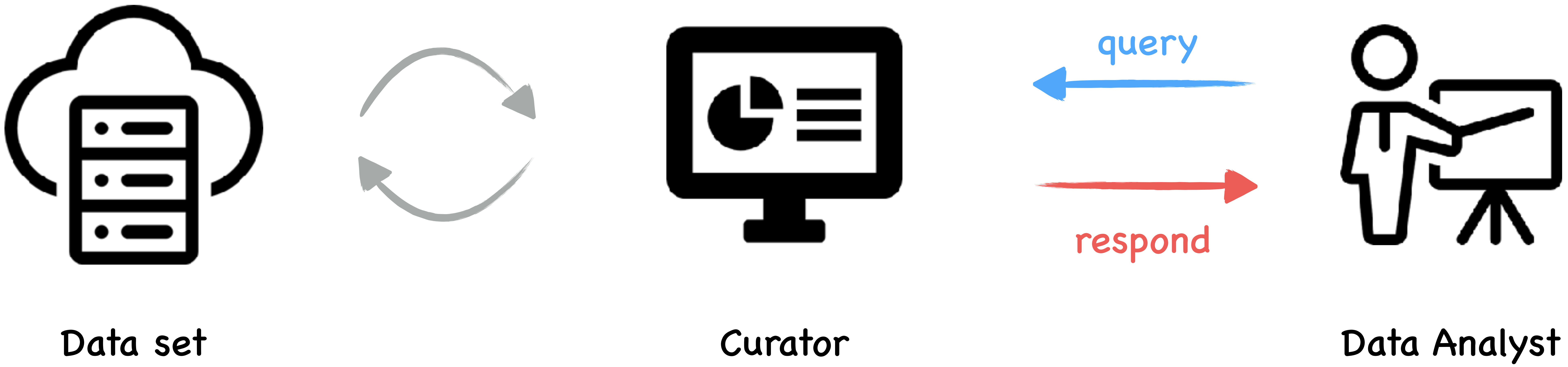

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$\boldsymbol{q}_i^\intercal = \underbrace{\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 1 \end{bmatrix}}_{n}$$

$$y_i = \boldsymbol{q}_i^\intercal \mathbf{x} + \boldsymbol{\Delta}$$

$y_i : \#$ of 1 in $\mathbf{x}$

Histogram Query as Linear Multiplication

Data set | Curator | Data Analyst

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} \boldsymbol{q}_1^\intercal \\ \boldsymbol{q}_2^\intercal \\ \vdots \\ \boldsymbol{q}_{T_n}^\intercal \end{bmatrix}$$

$$\boldsymbol{y} = \mathbf{Q}\mathbf{x} + \boldsymbol{\Delta}$$

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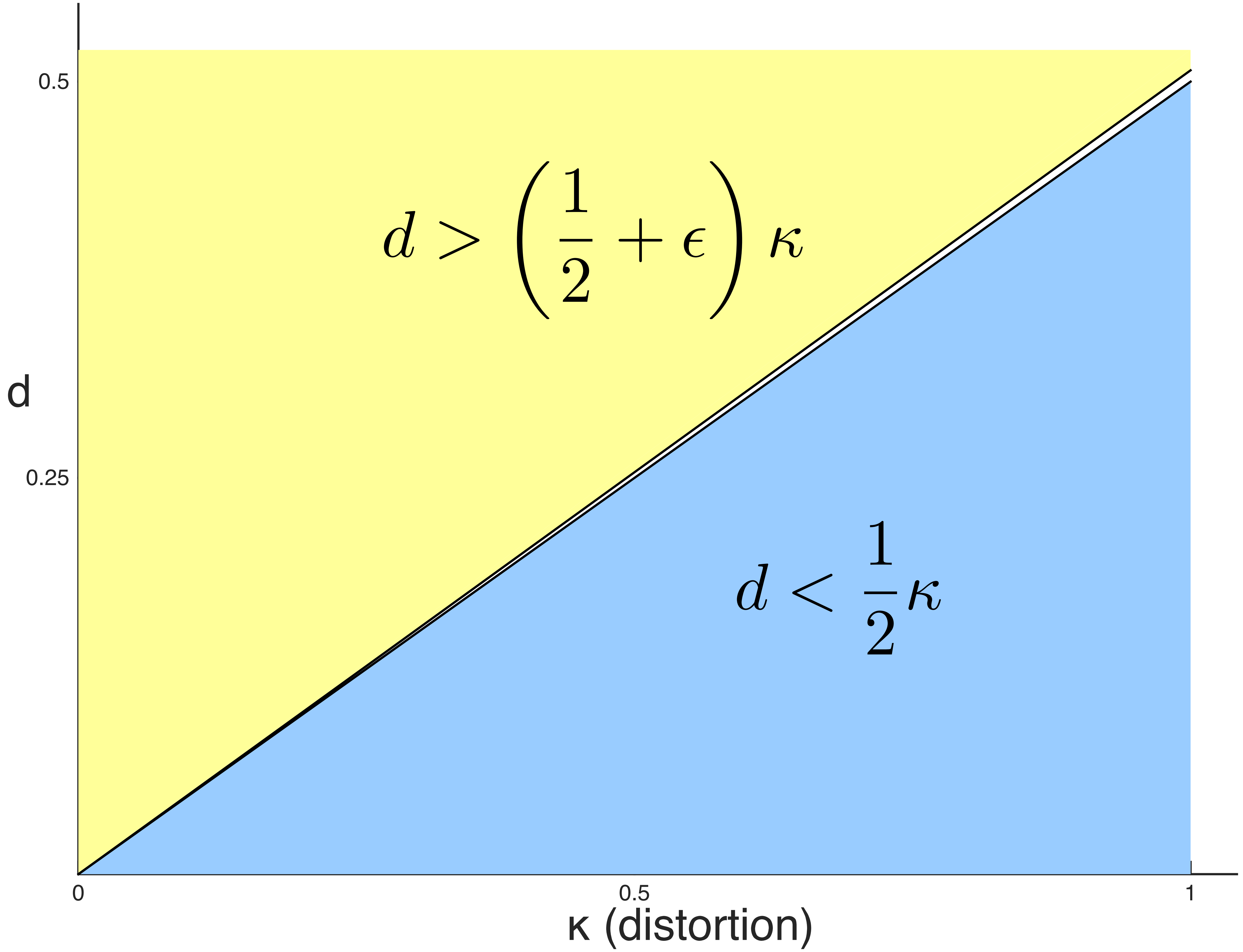
- Query complexity $T_n^*(k_n, \delta_n)$: minimum number of queries required to extract data set within distortion k_n, under noise level δ_n

Main Result

$$\delta_n = \Theta\left(n^d\right), \; k_n = \Theta\left(n^\kappa\right)$$



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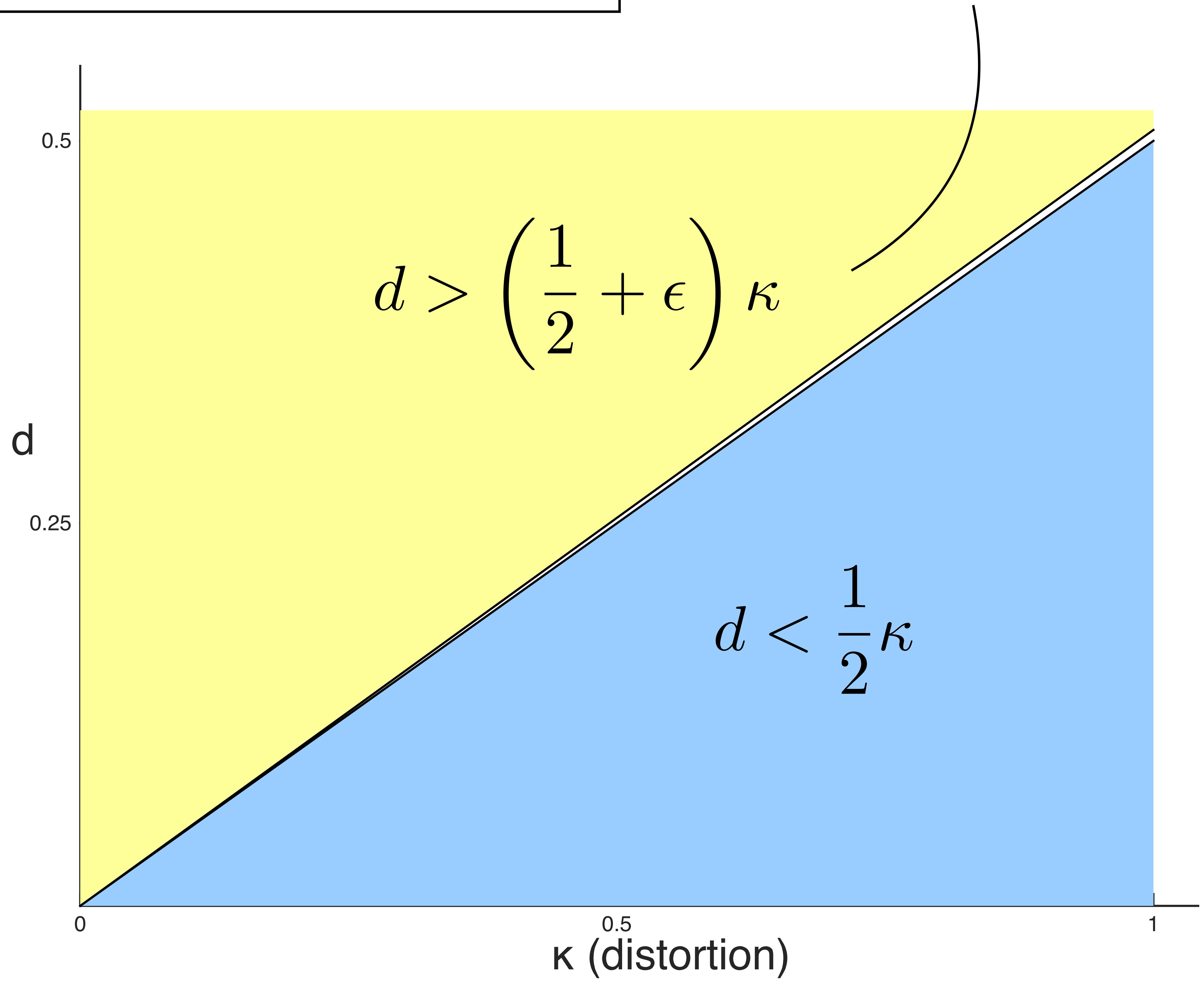

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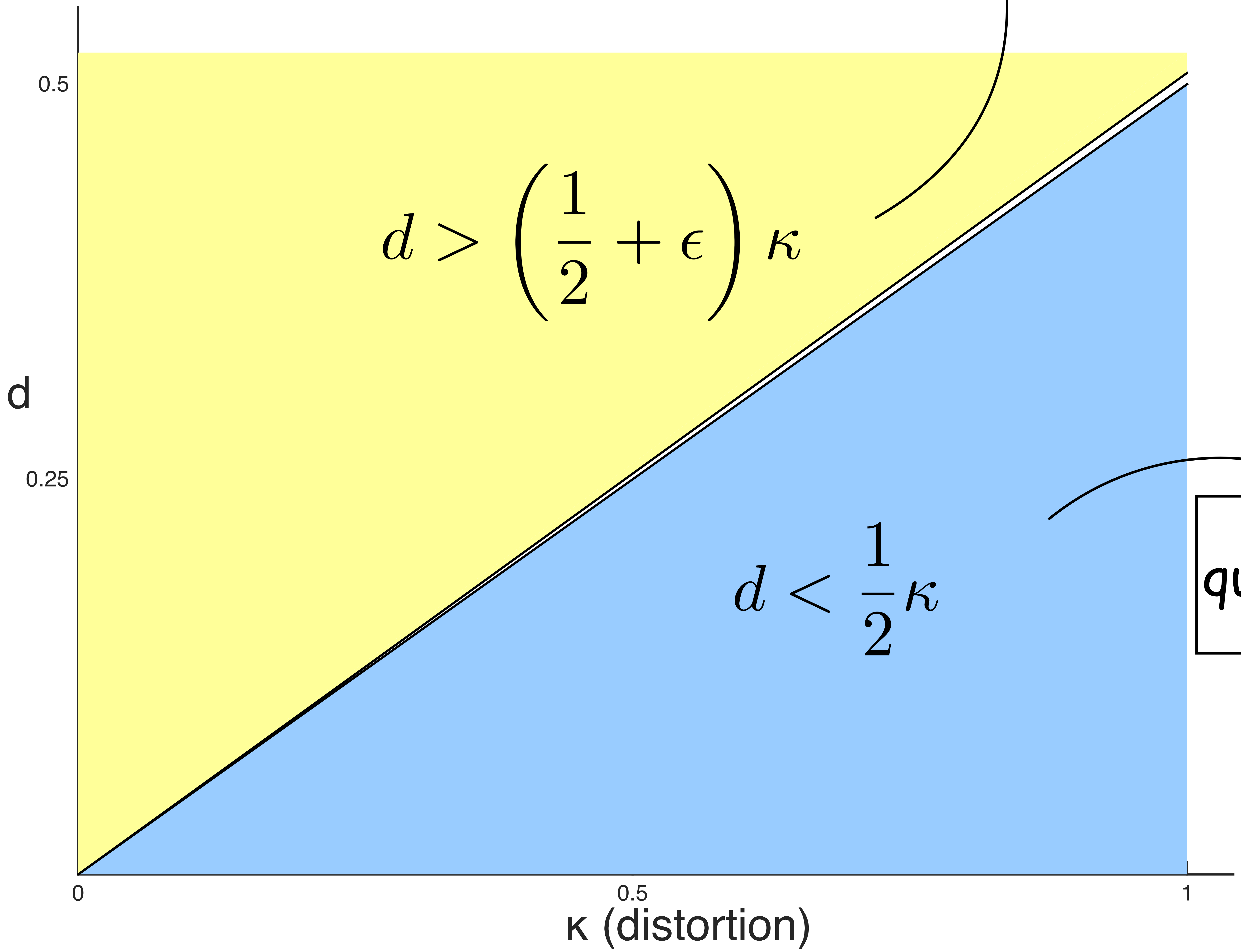


$d > \left(\frac{1}{2} + \epsilon\right)\kappa$

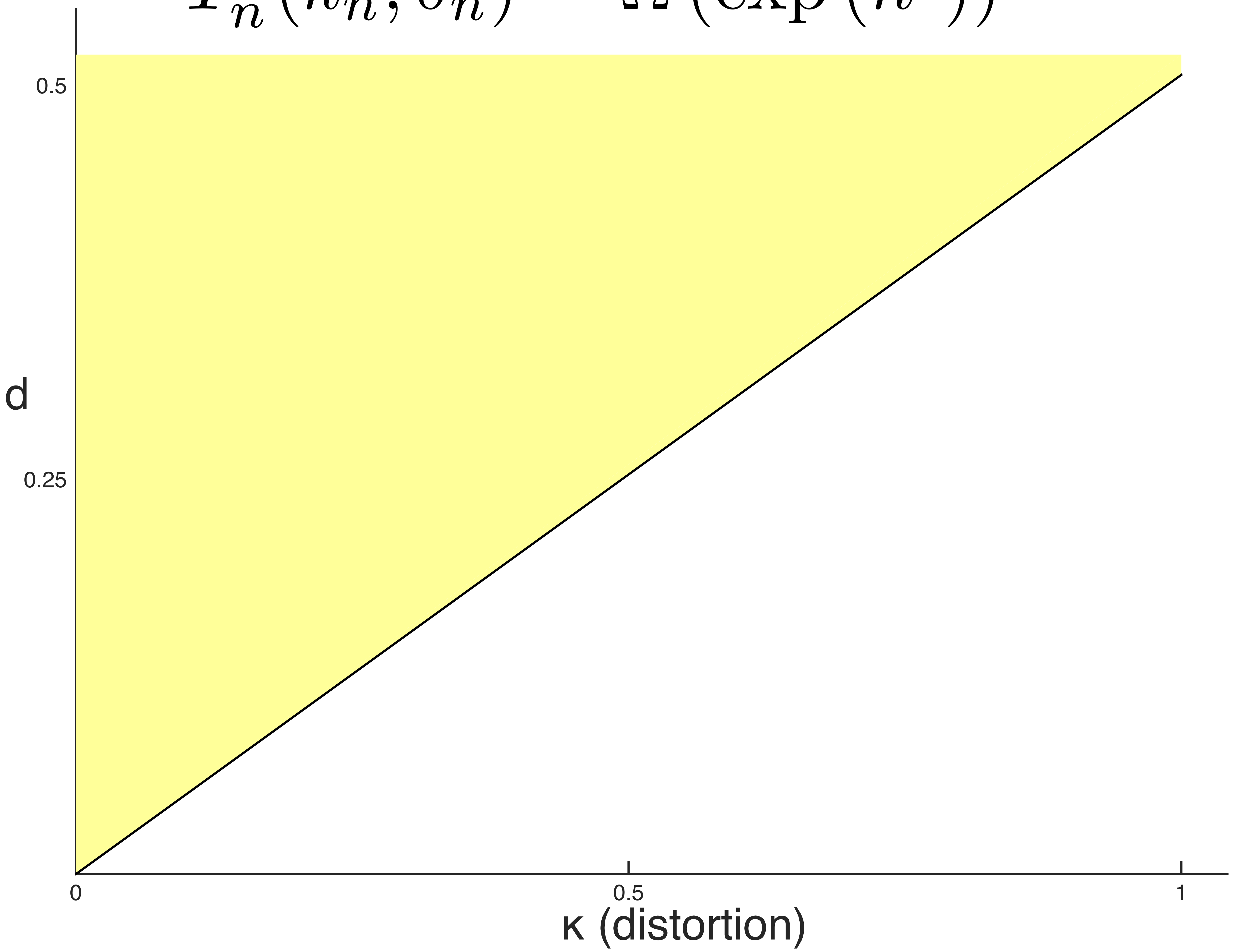
$d < \frac{1}{2}\kappa$

query complexity: sub-linear $\Theta\left(\frac{n}{\log n}\right)$

Regime 1: Impossibility of Polynomial Query Complexity

- **Regime 1:** $d > (\frac{1}{2} + \epsilon)\kappa, \text{ for any } \epsilon > 0$ (the noise is too large)

$$T_n^*(k_n, \delta_n) = \Omega\left(\exp\left(n^{\epsilon}\right)\right)$$



Impossibility of Polynomial Query

- Proof idea : without sufficient number of queries, there exists more than one possible data set which are consistent with the response.

Impossibility of Polynomial Query

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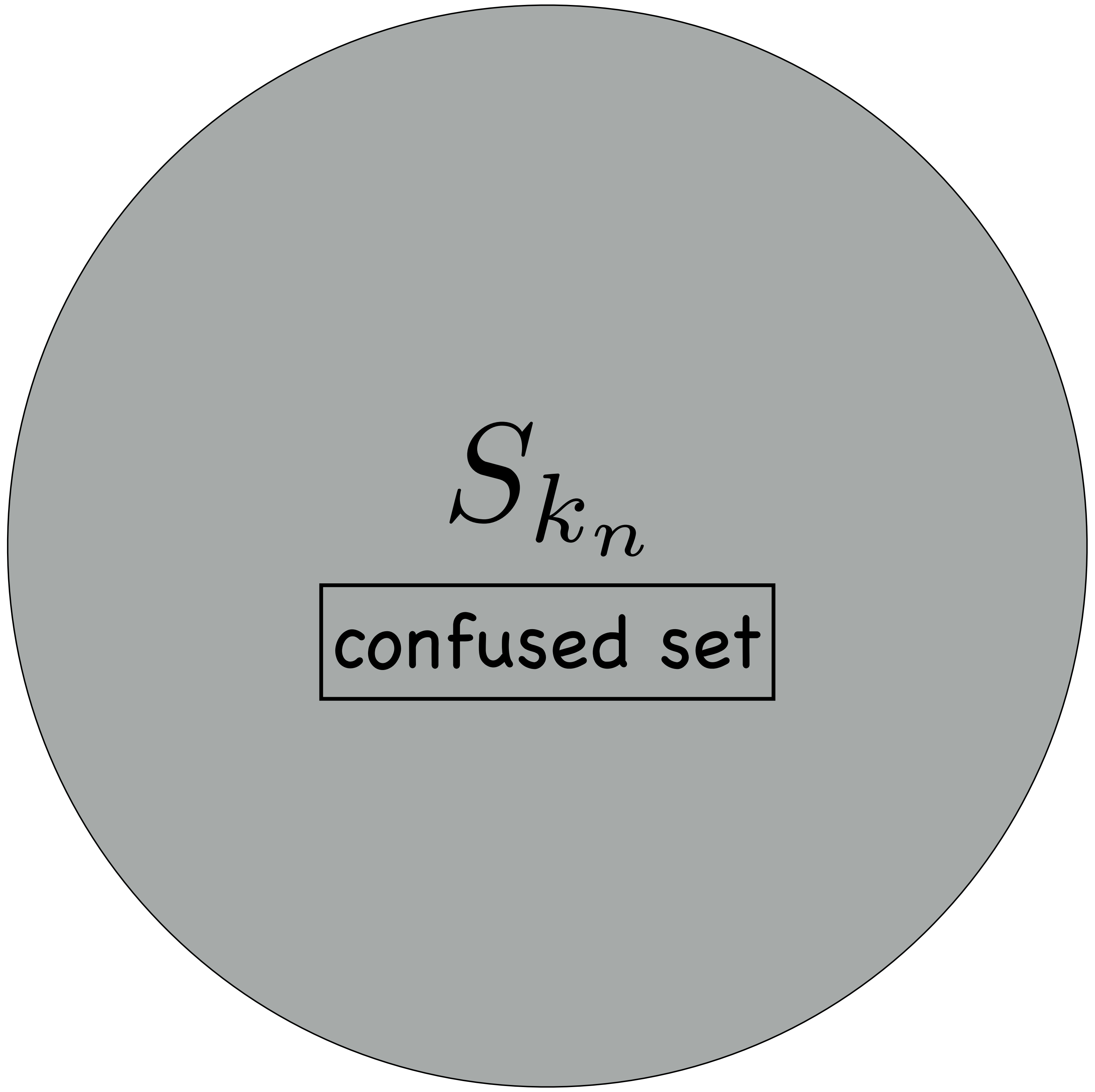
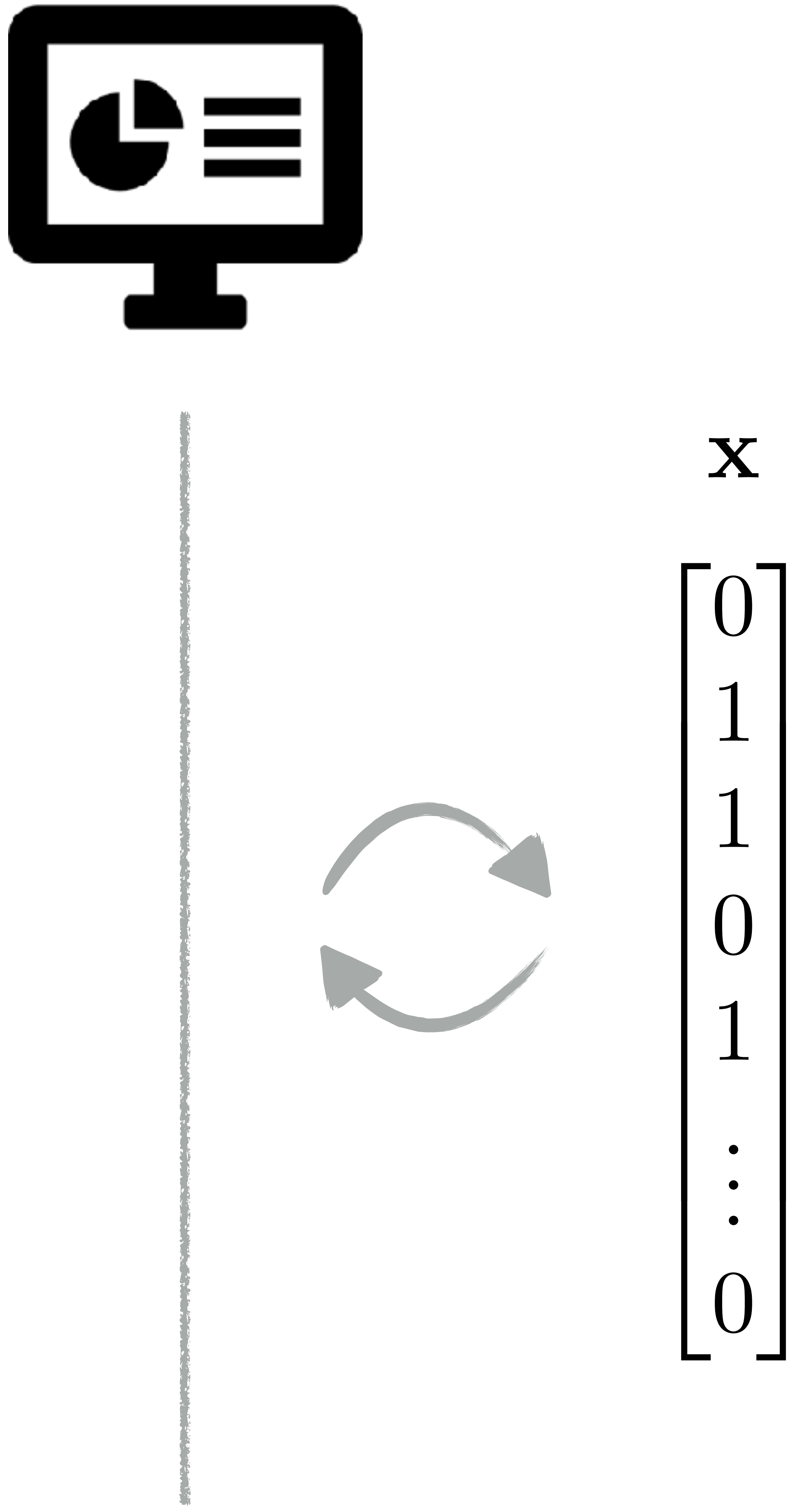
$$S_{k_n} \triangleq \{(\mathbf{x}, \tilde{\mathbf{x}}) \mid \mathbf{x}, \tilde{\mathbf{x}} \in \{0,1\}^n, \|\mathbf{x} - \tilde{\mathbf{x}}\|_1 = k_n, \|\mathbf{x}\|_1 = \|\tilde{\mathbf{x}}\|_1\}$$





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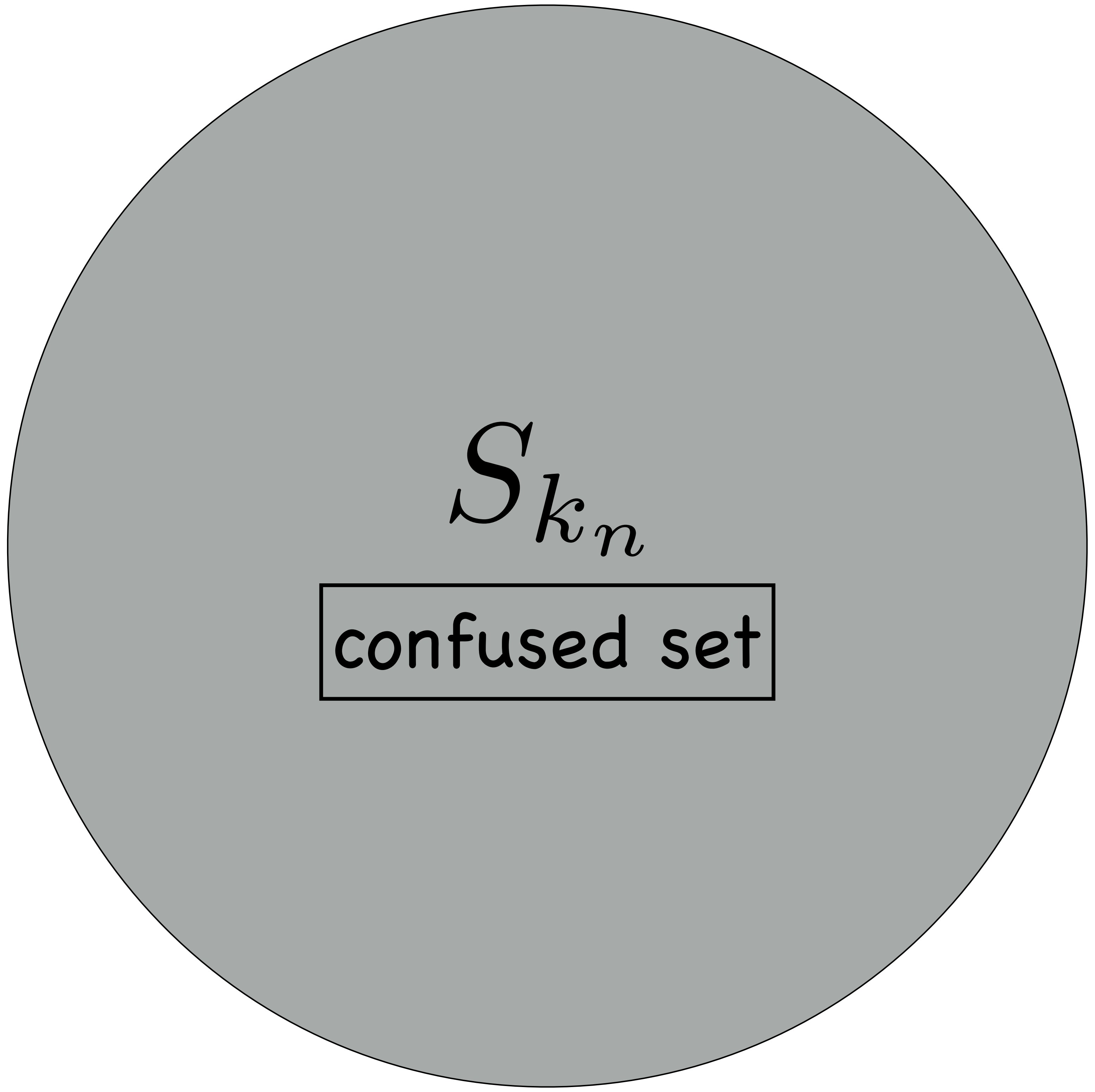
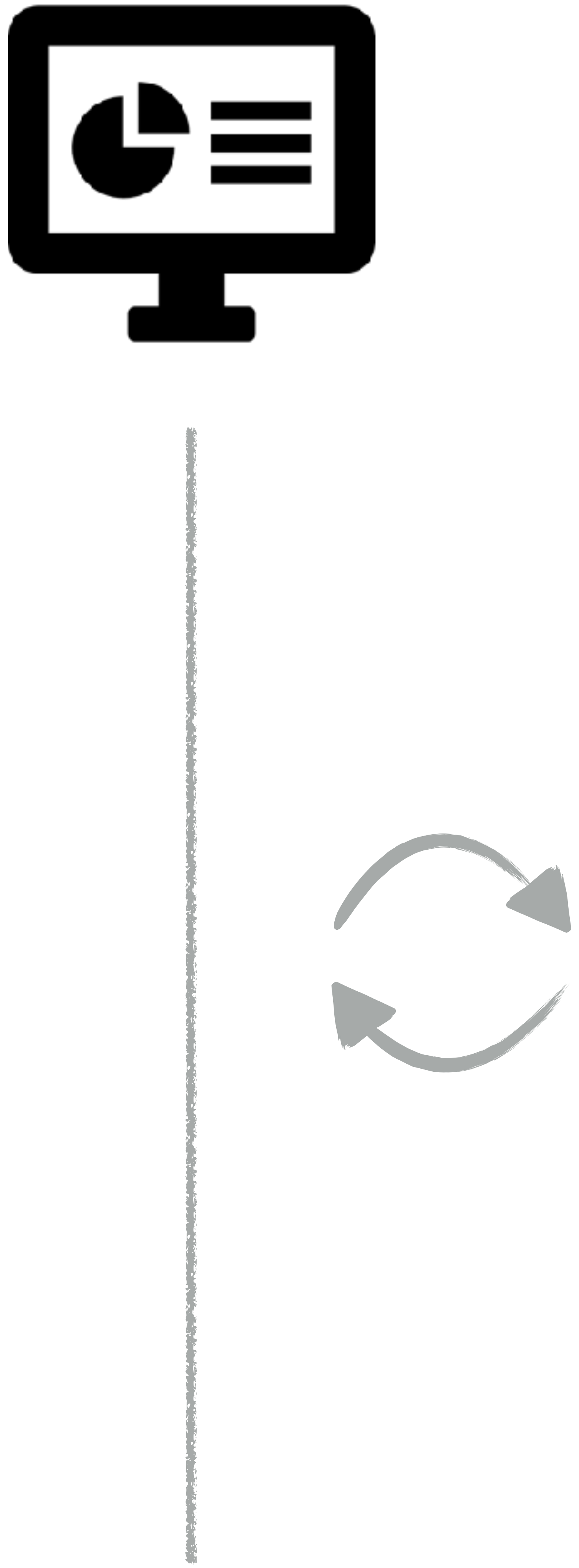
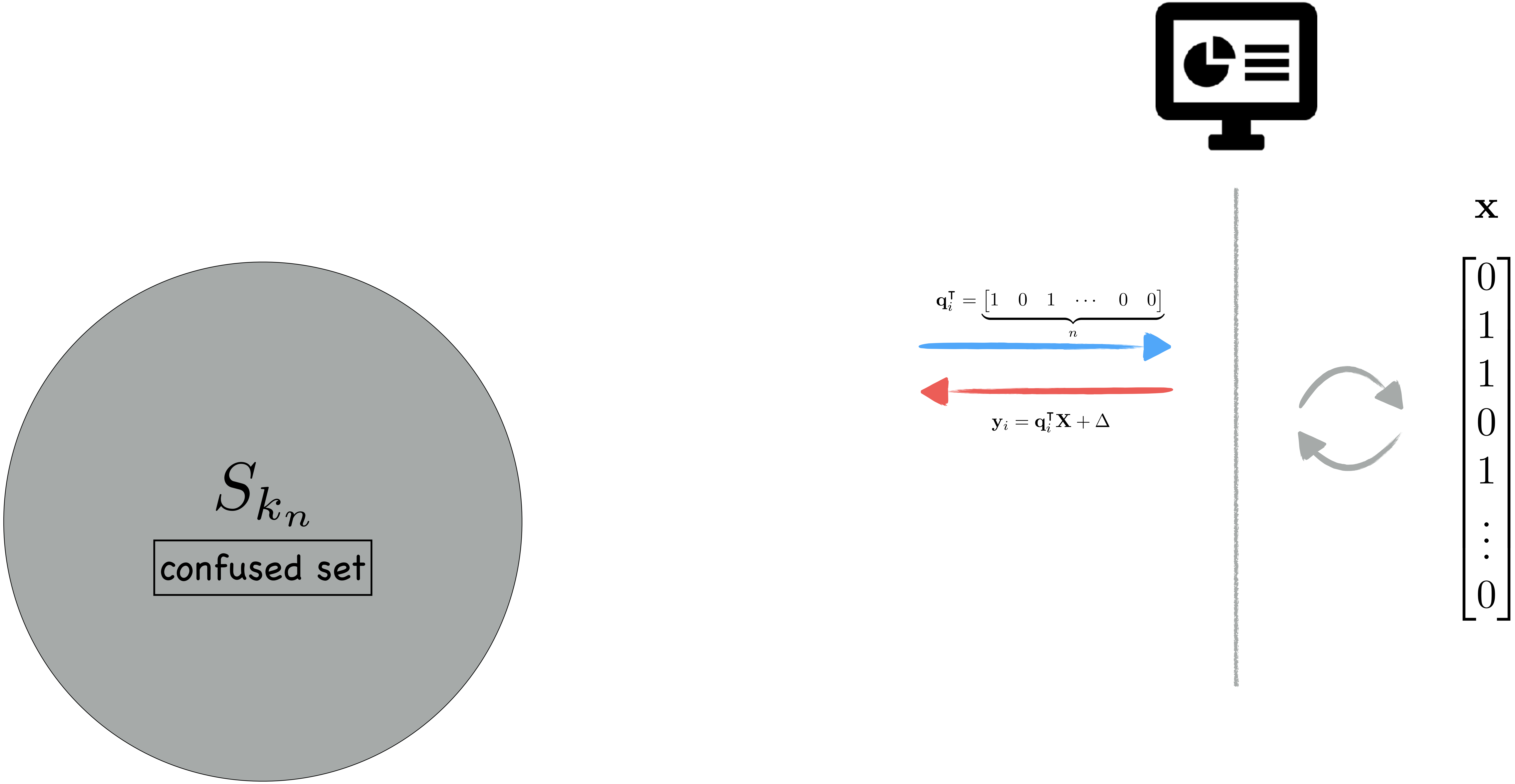




$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Impossibility of Polynomial Query



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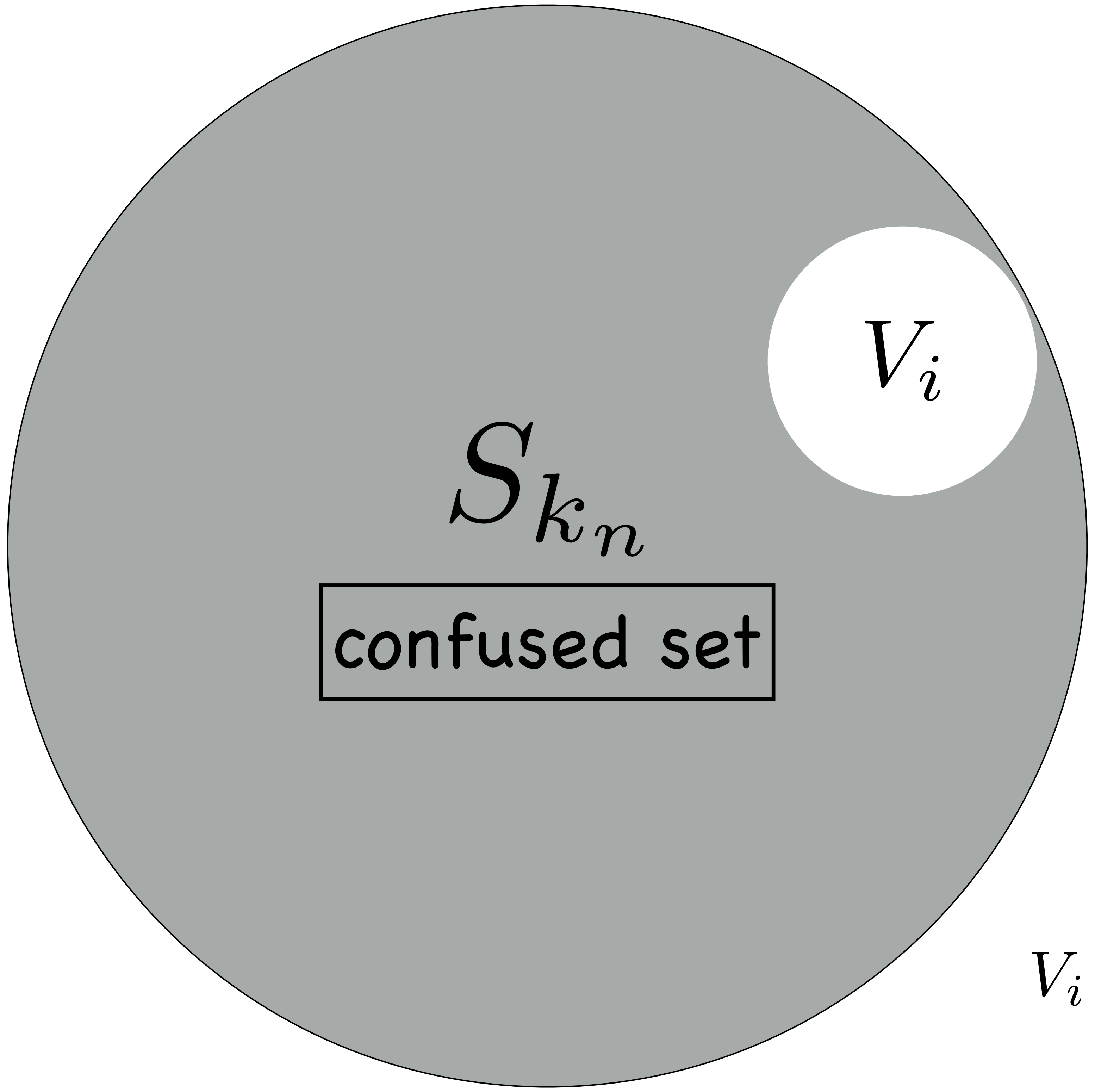

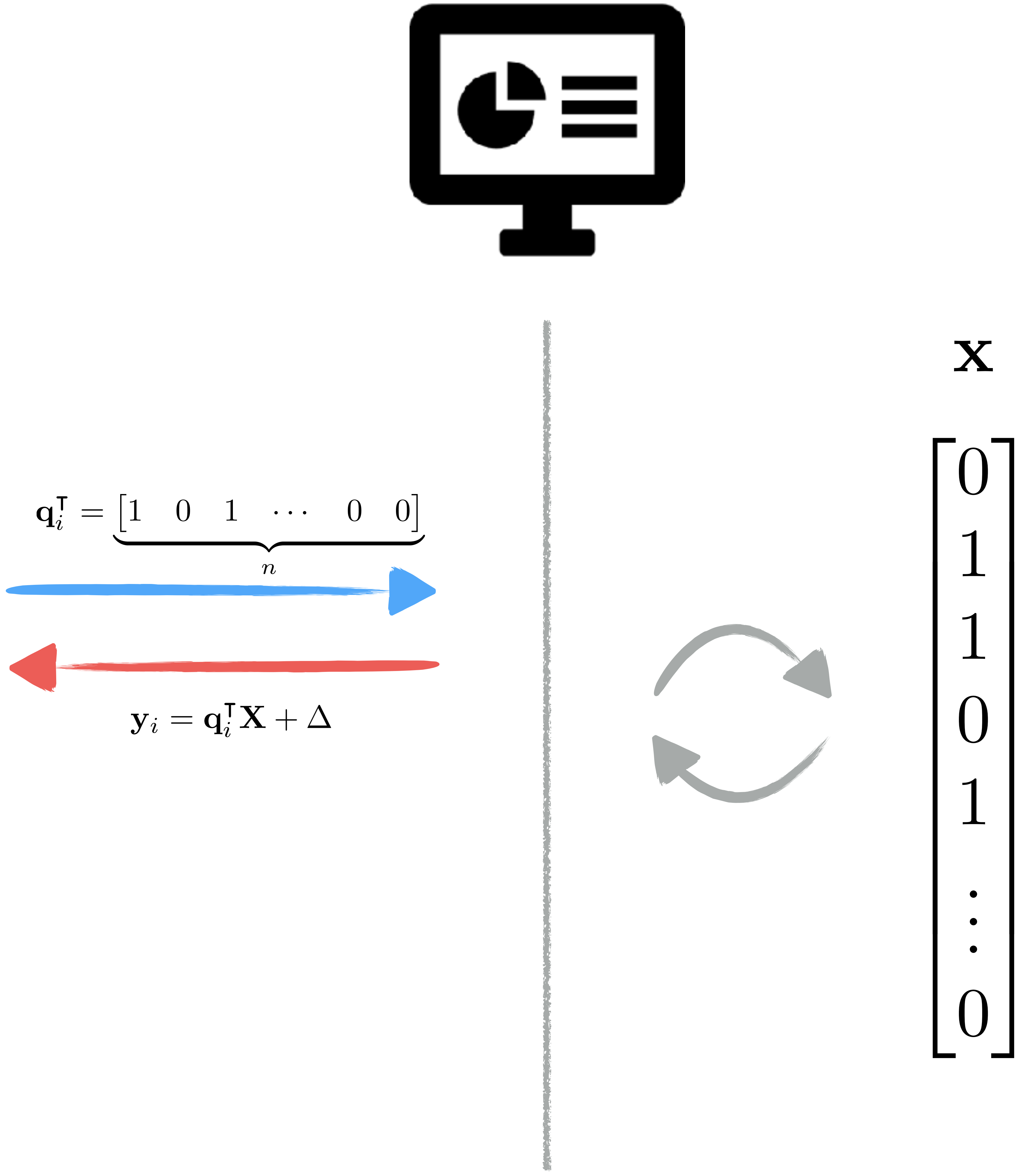

$$V_i \triangleq \{(\mathbf{x}, \tilde{\mathbf{x}}) \in S_{k_n} \mid |\mathbf{q}_i \cdot (\mathbf{x} - \tilde{\mathbf{x}})| > \delta_n\}.$$

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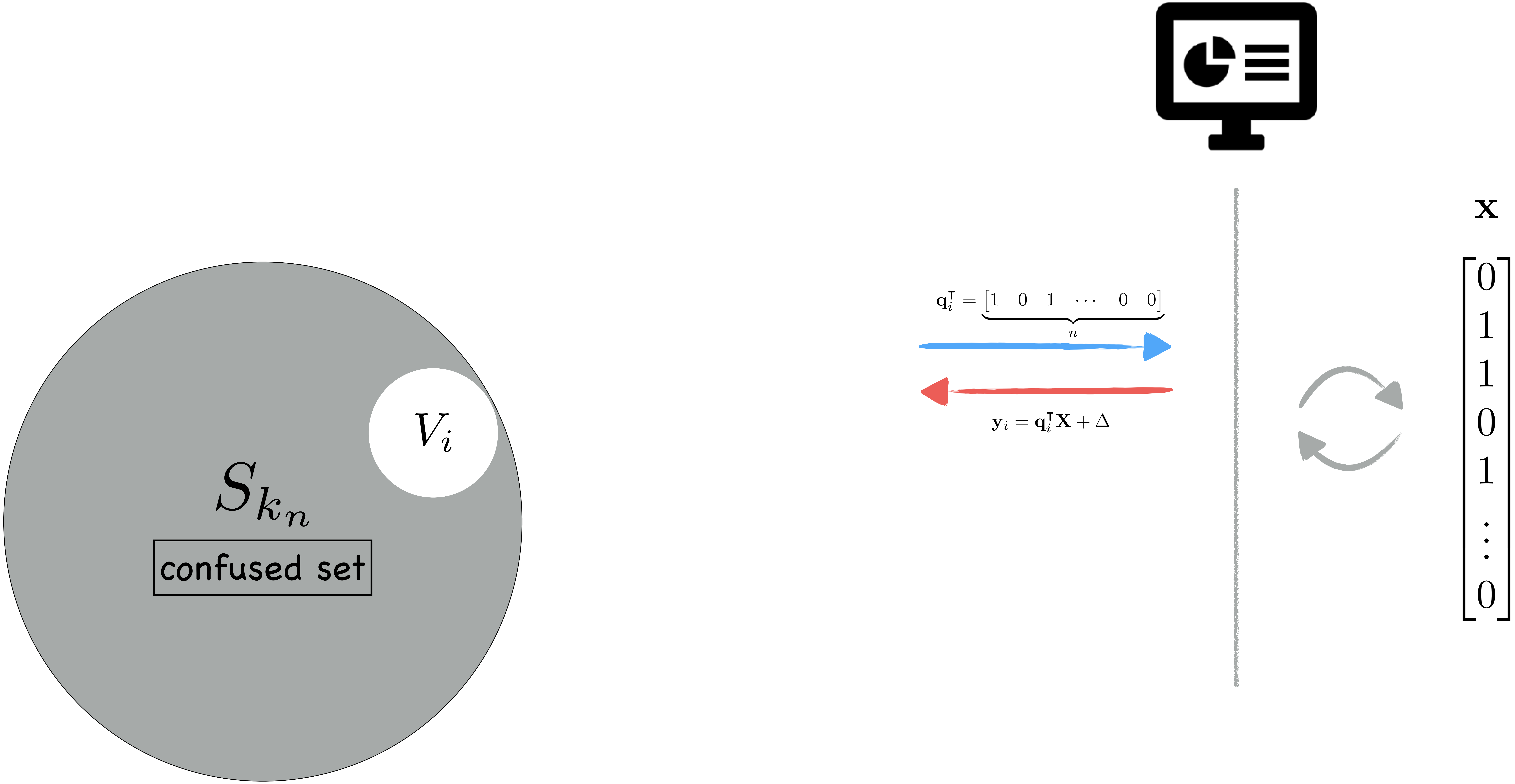

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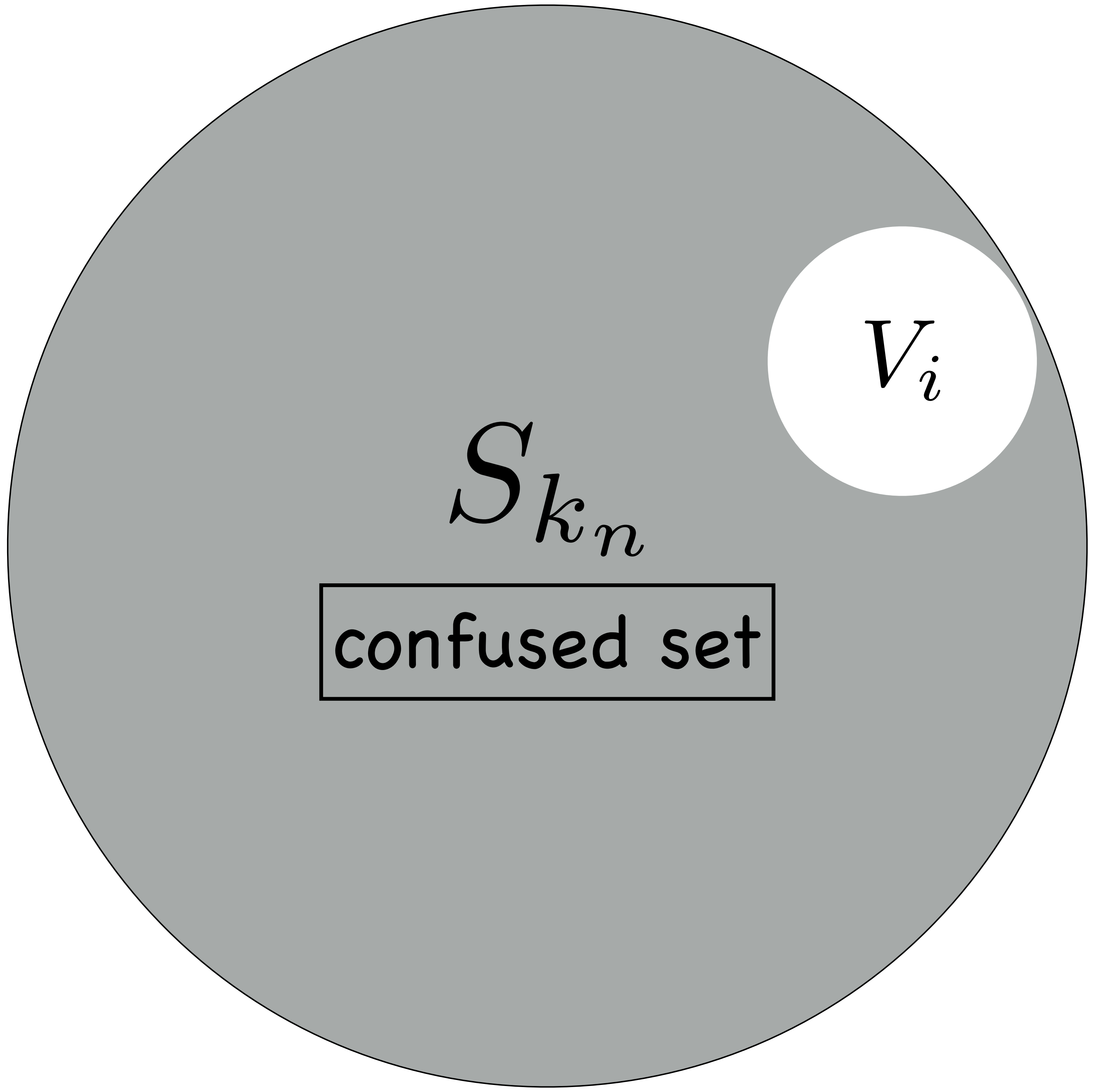

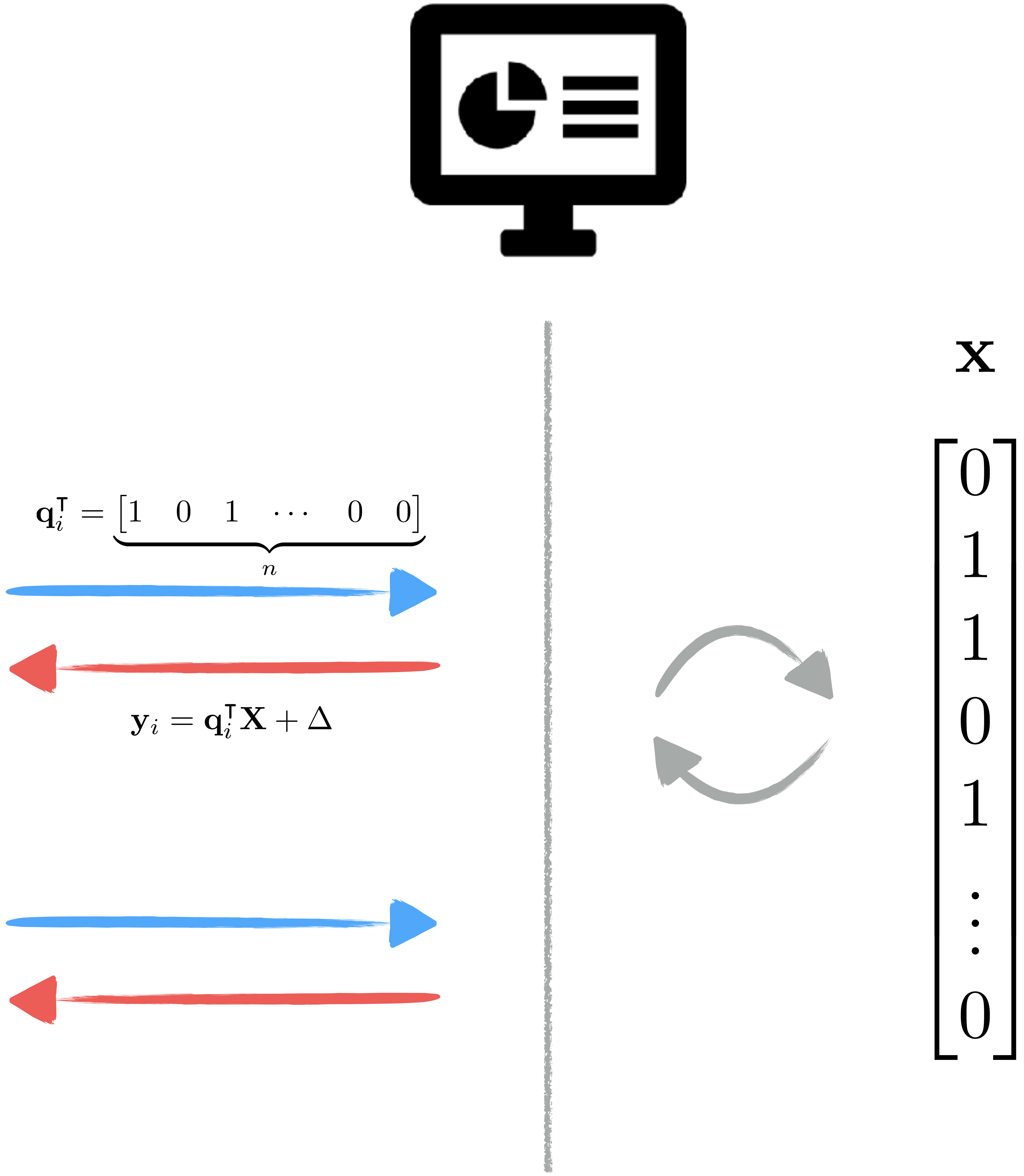

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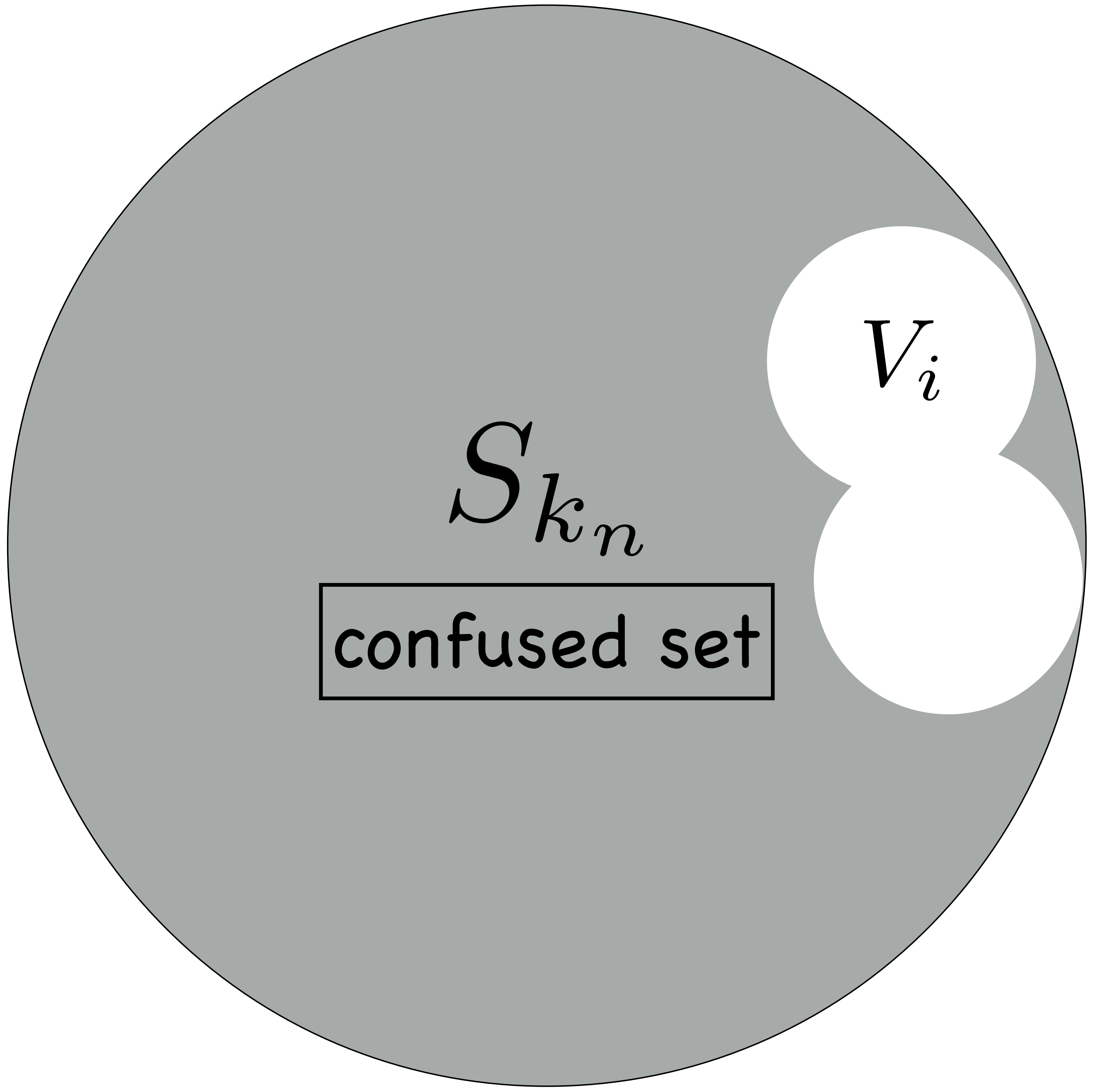

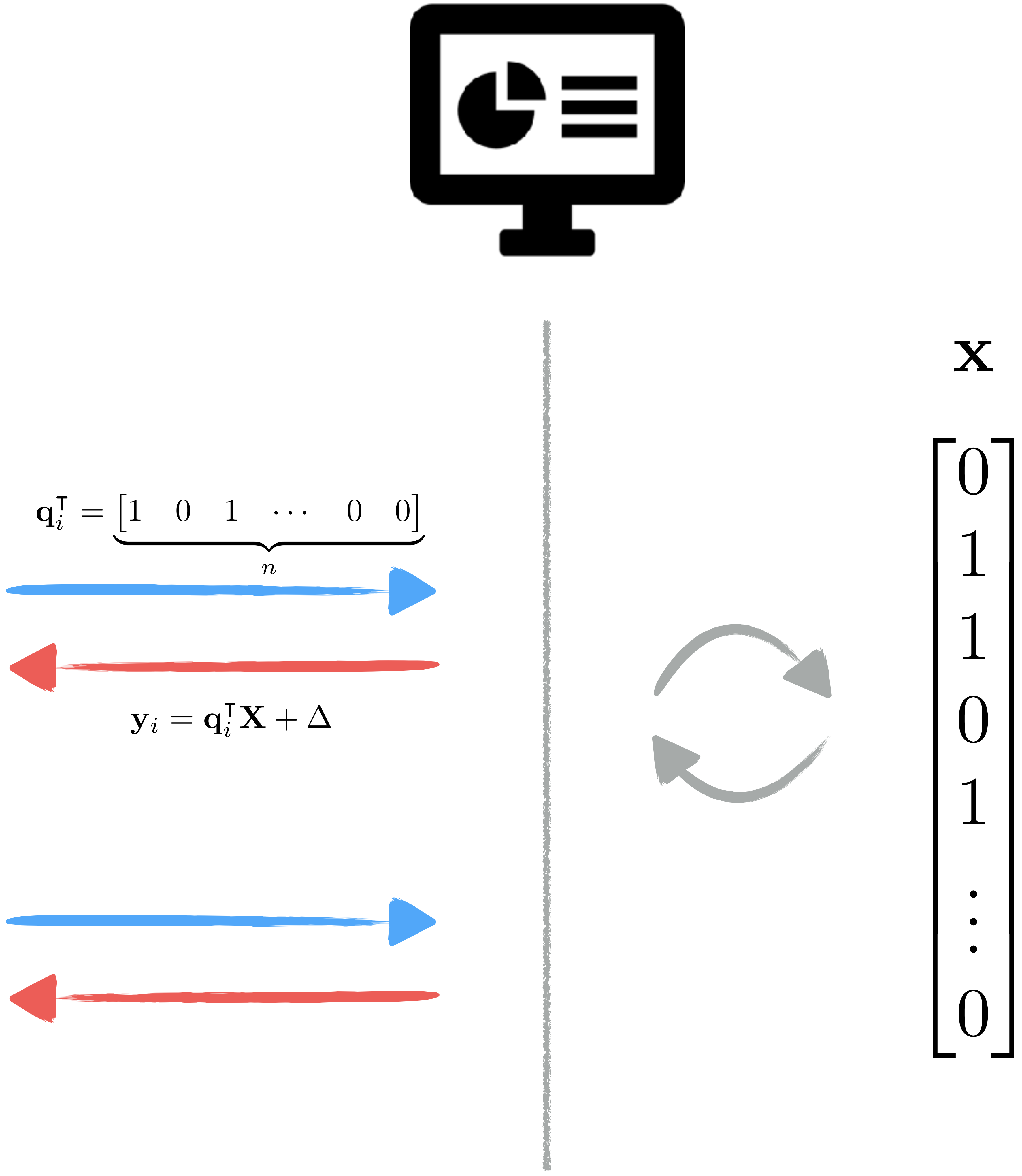

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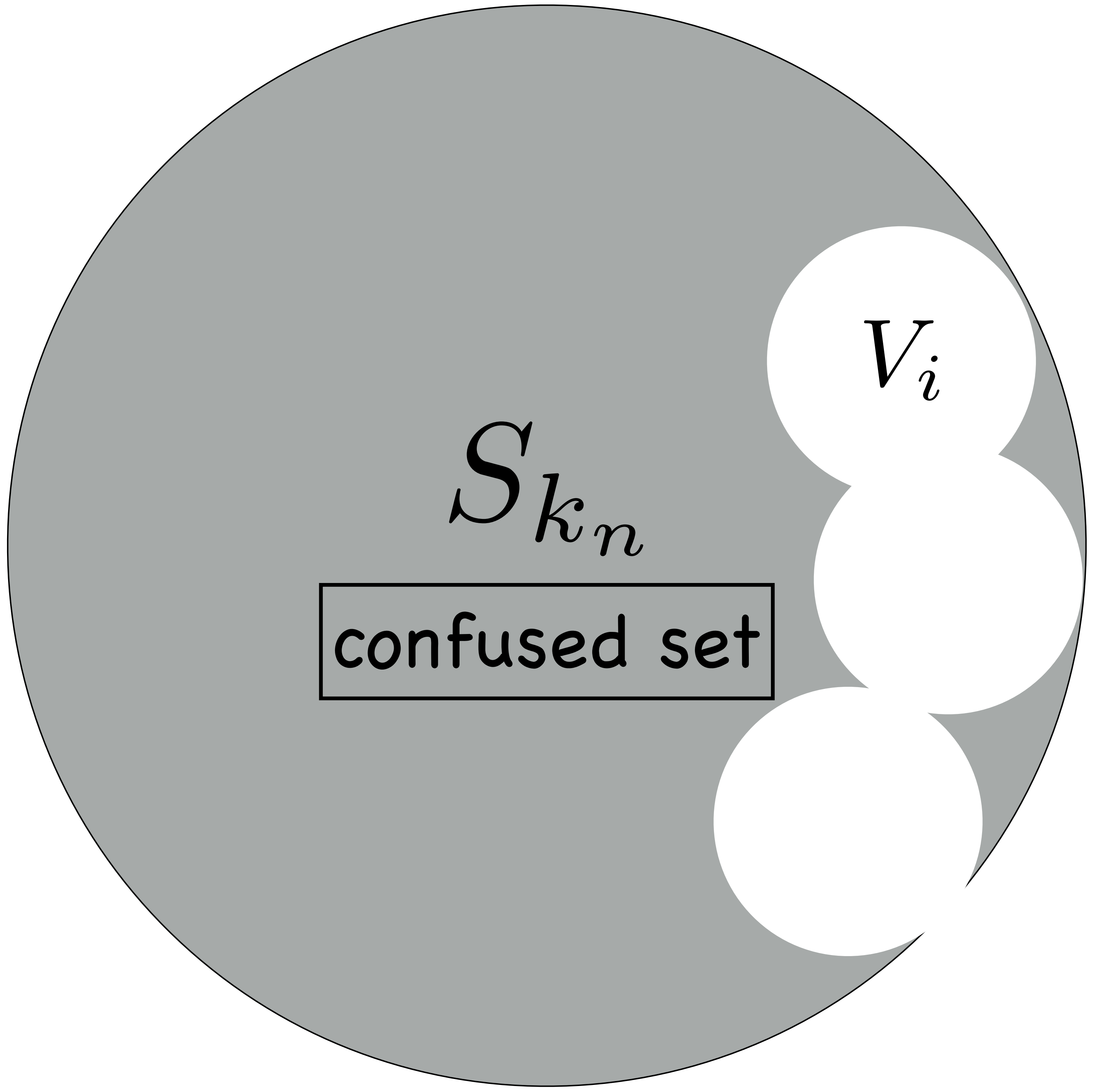

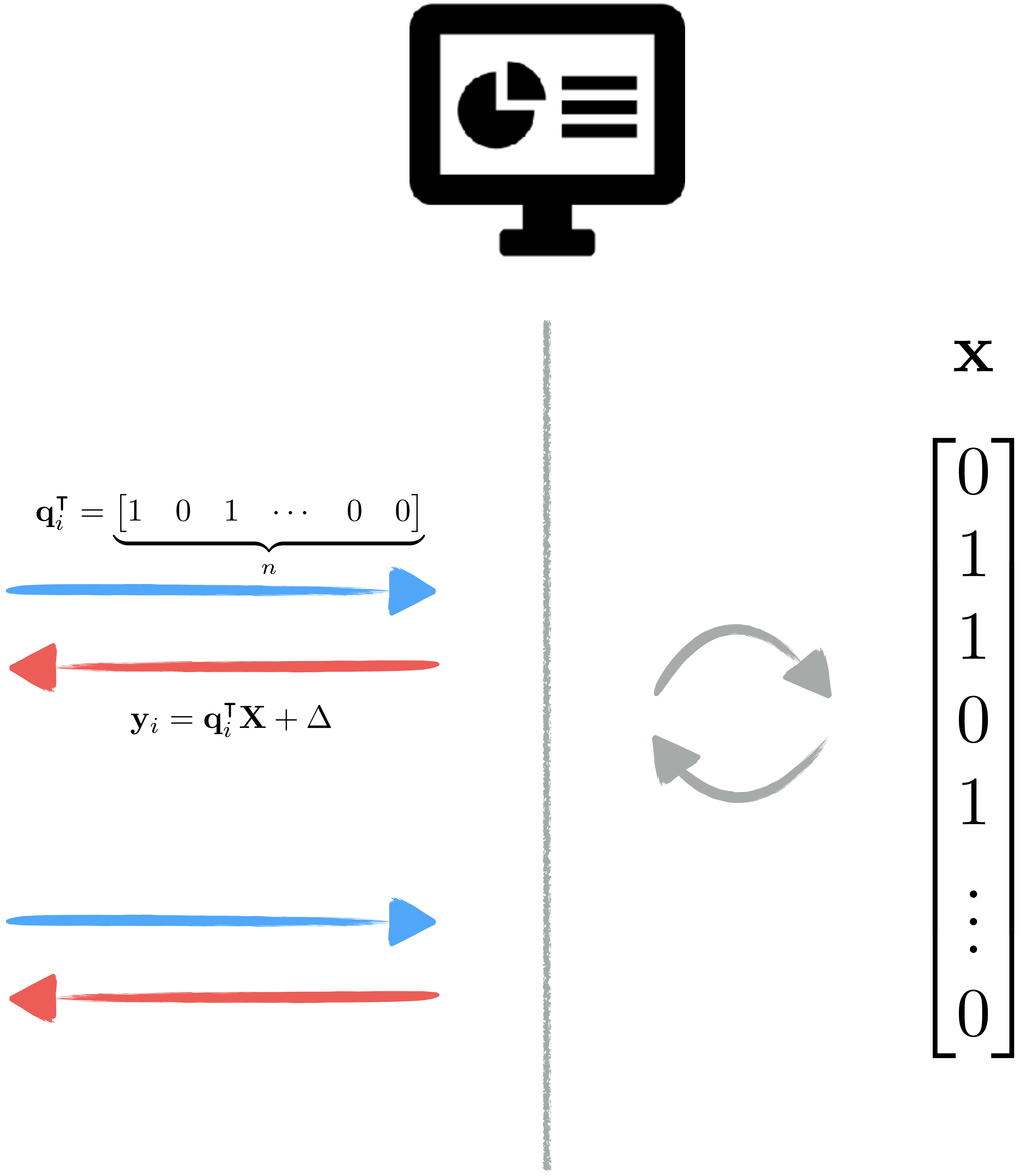

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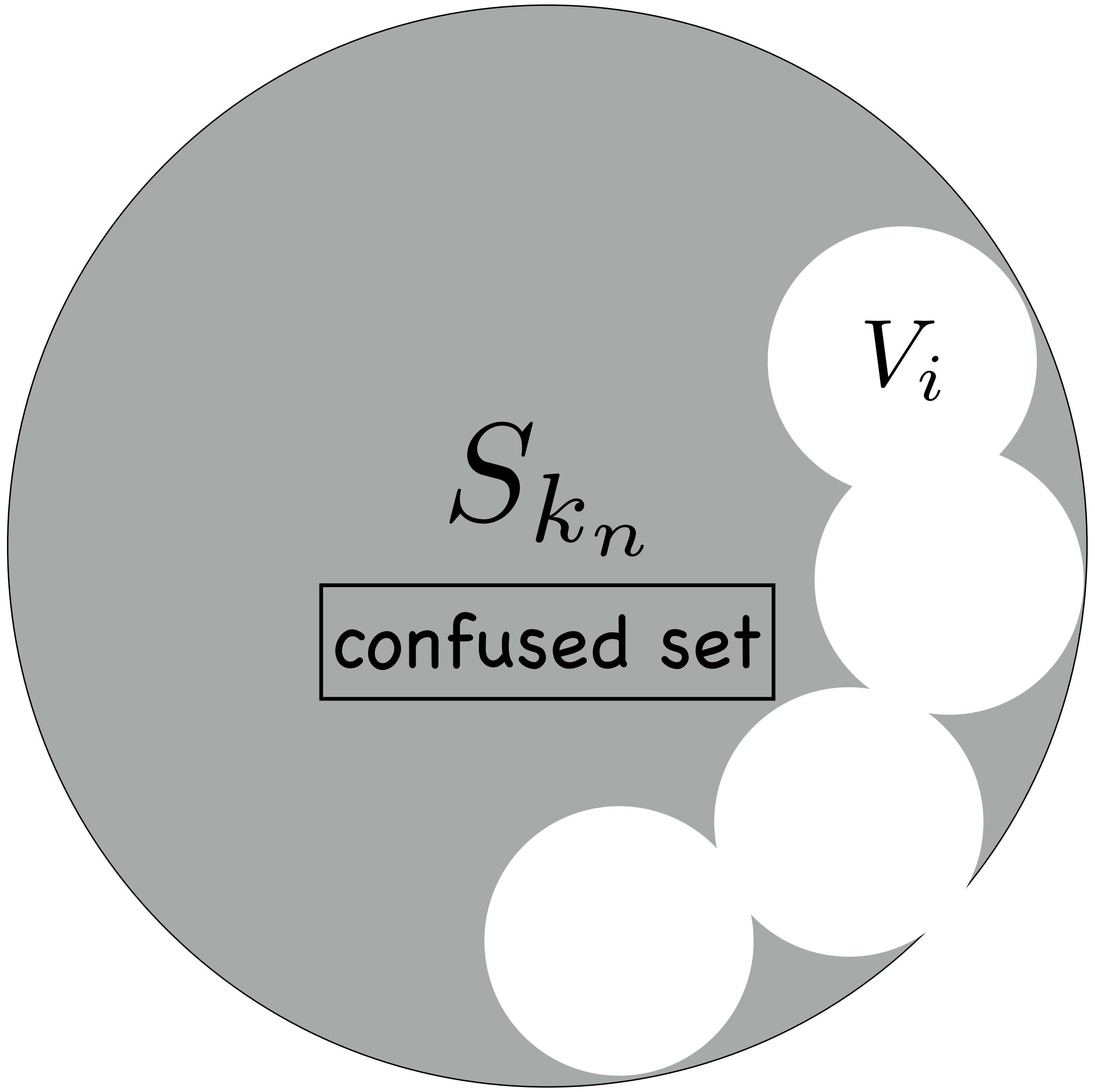

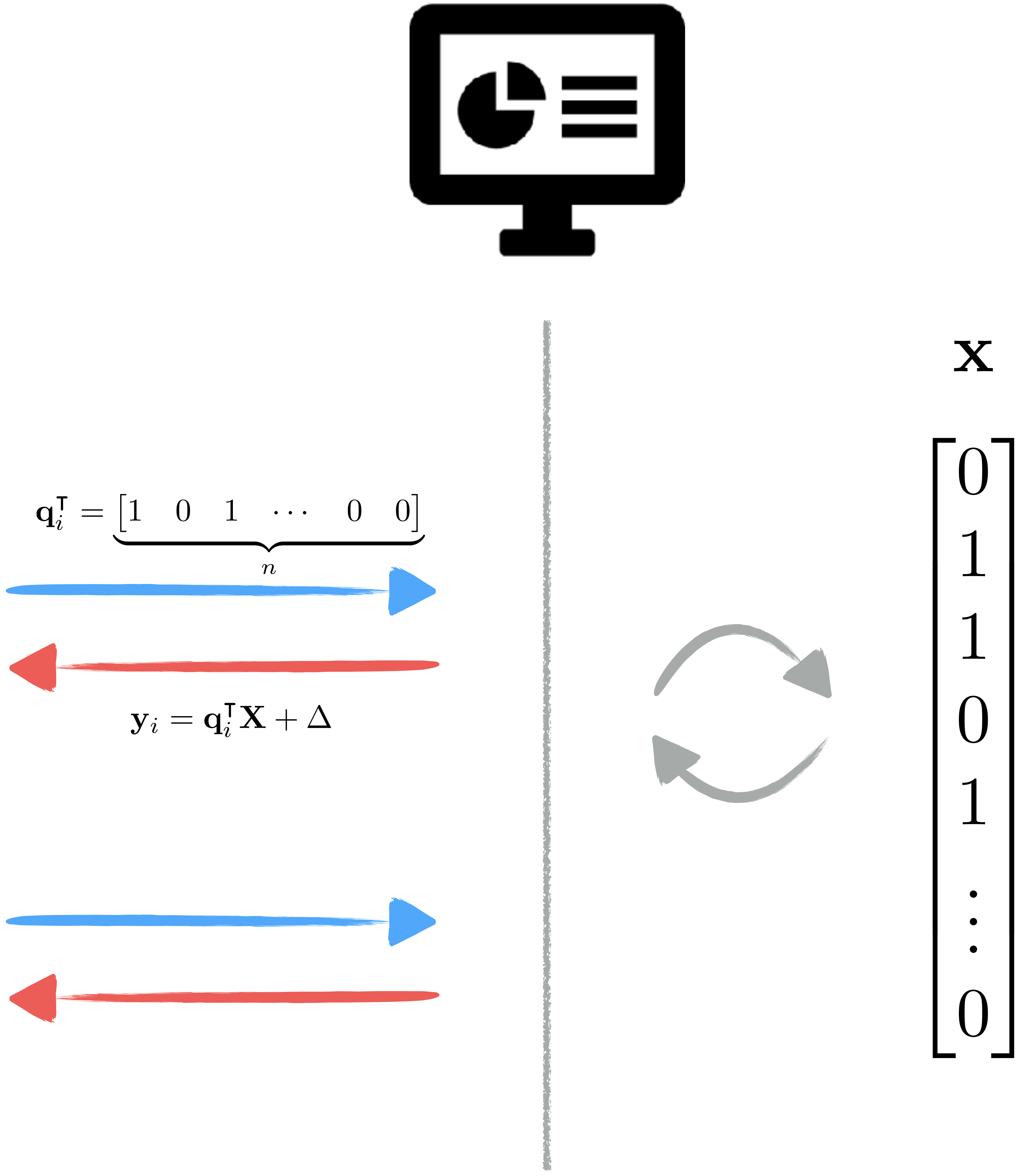

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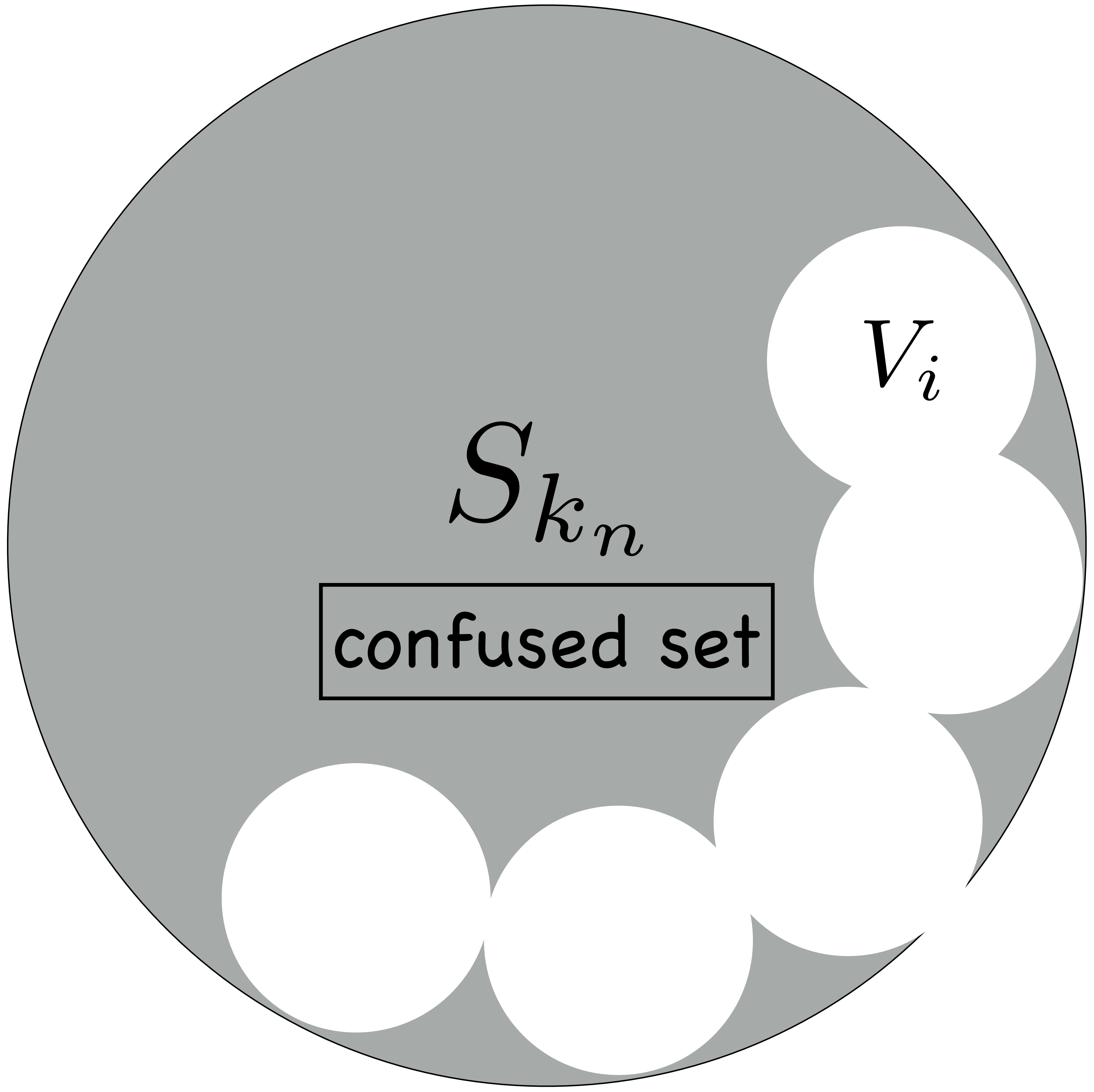

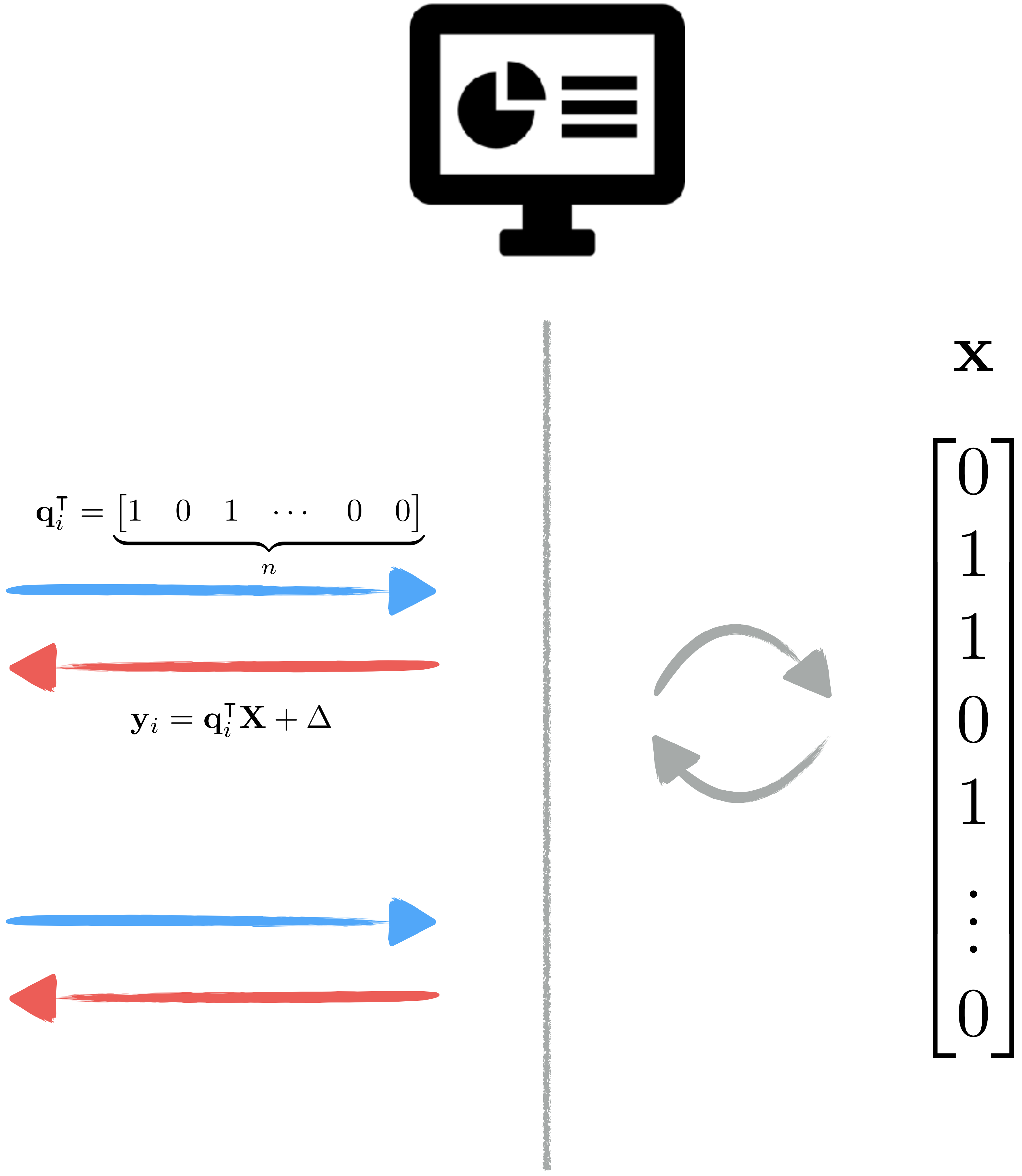

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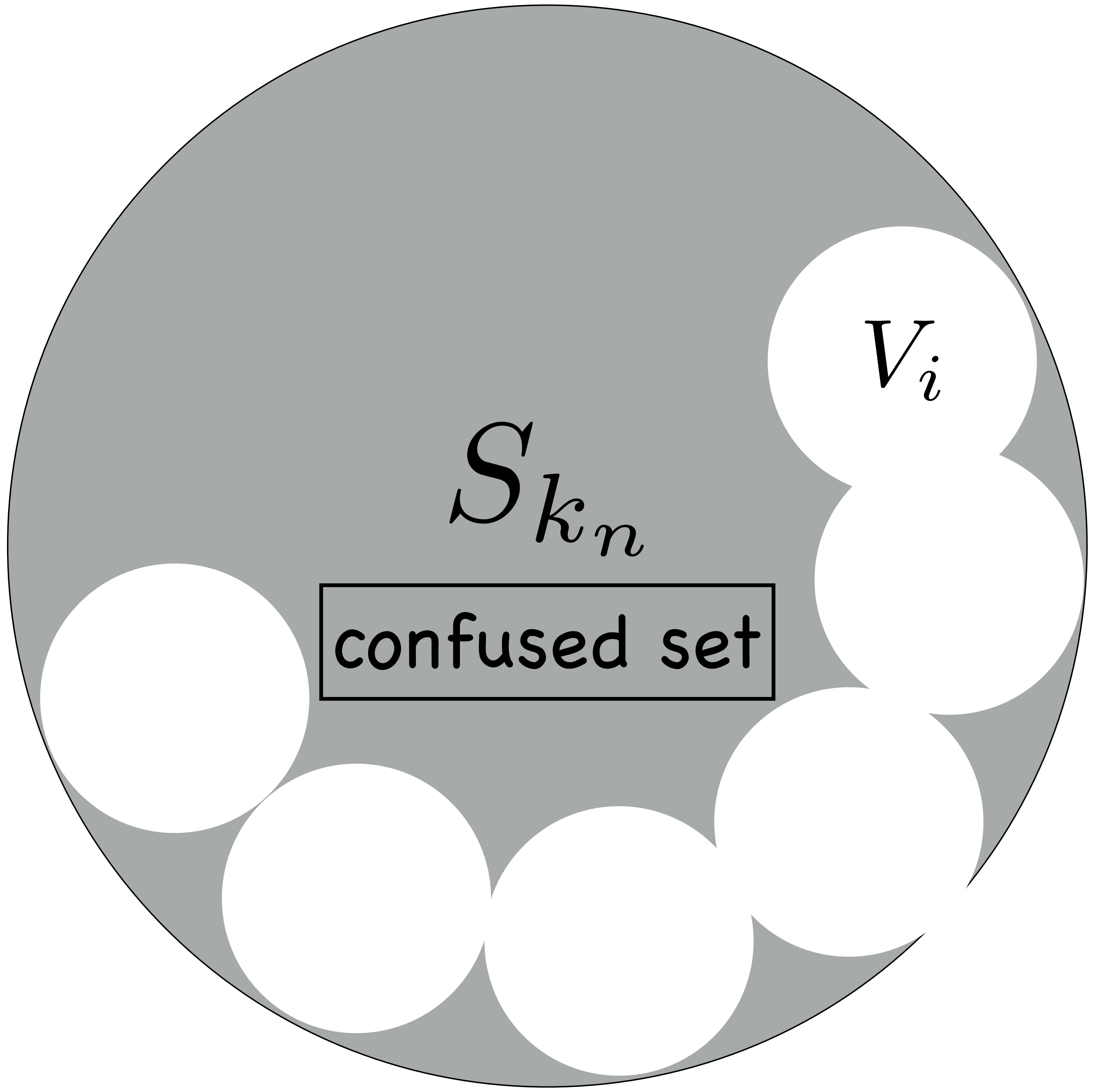

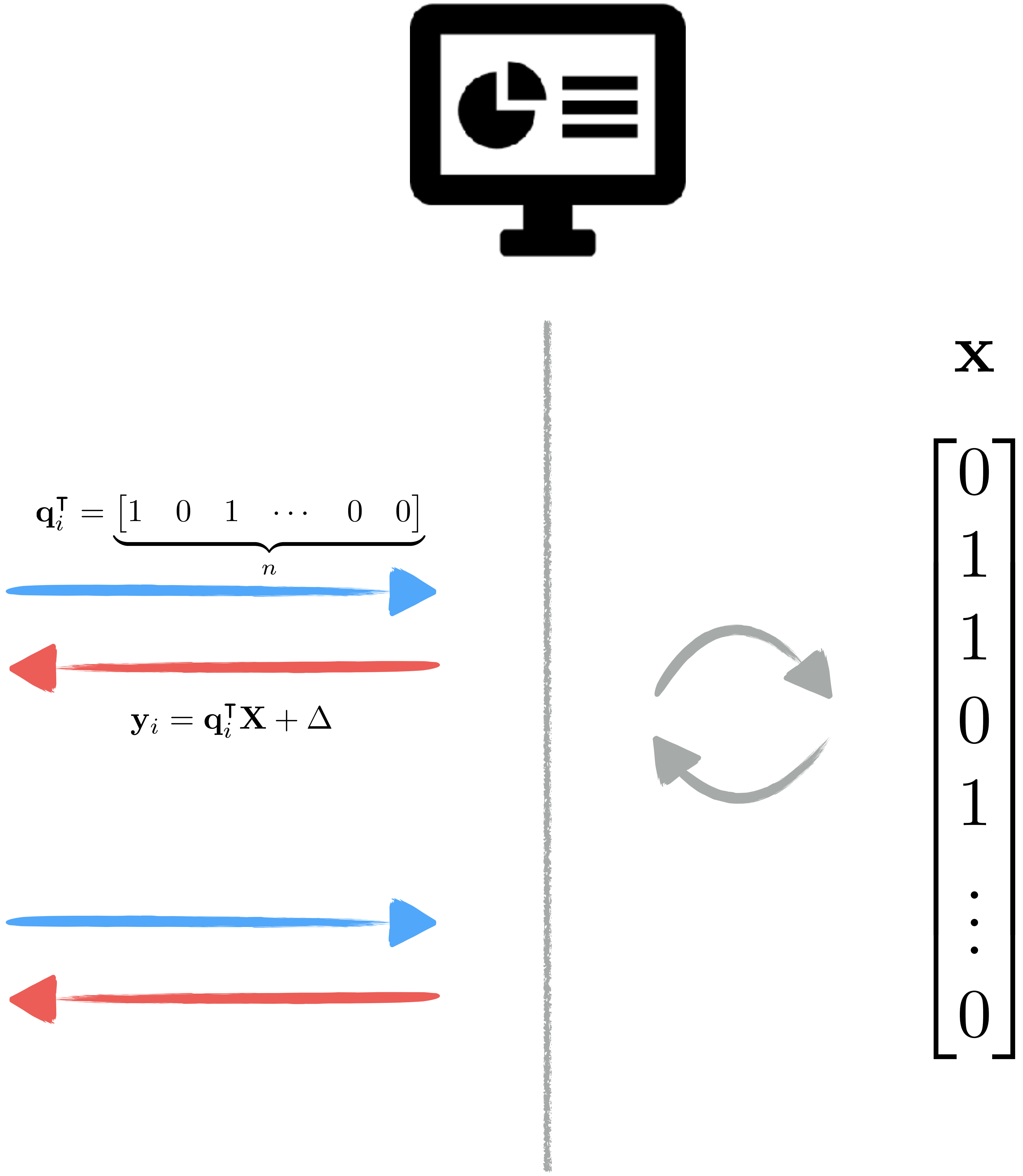

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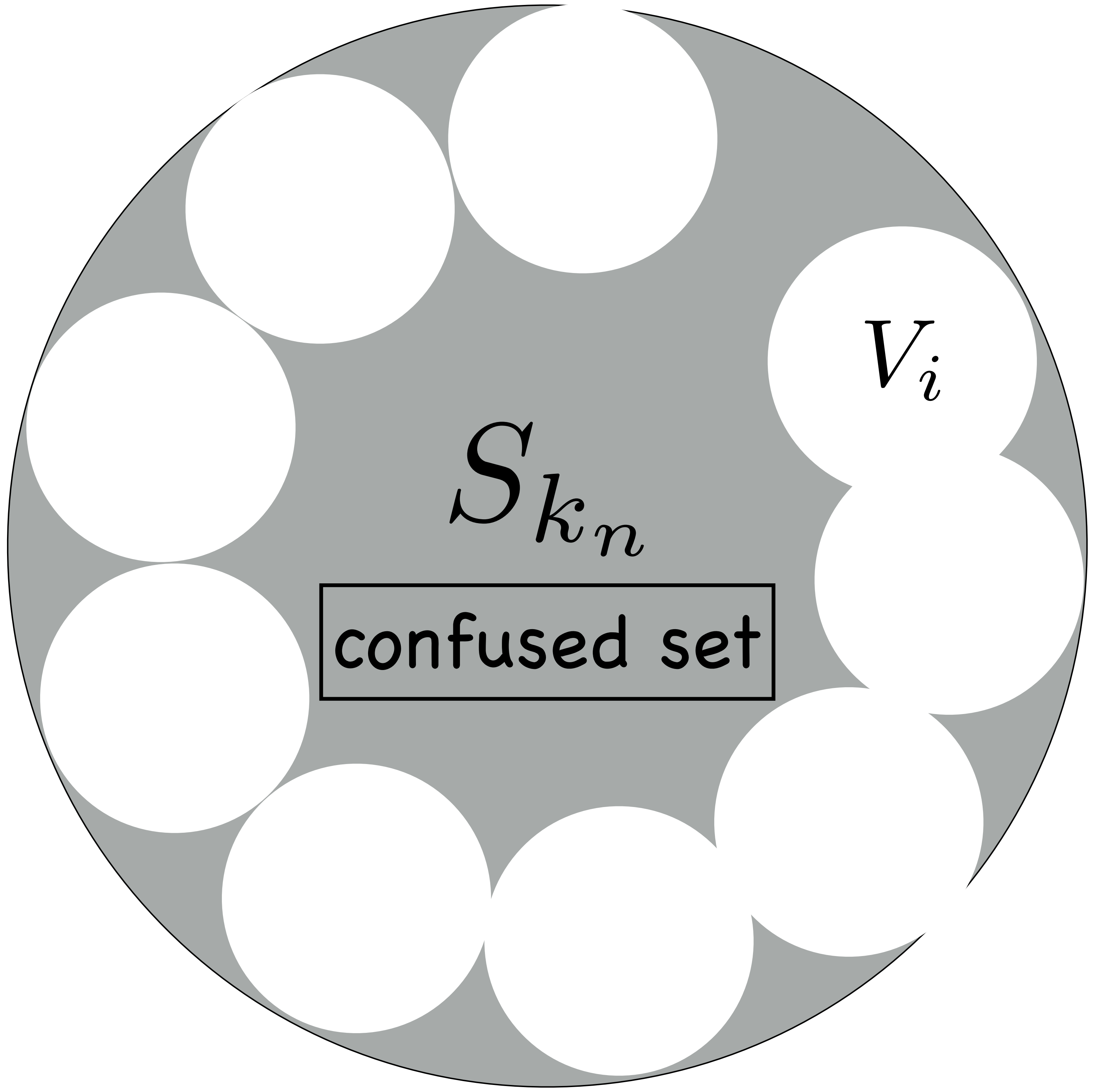
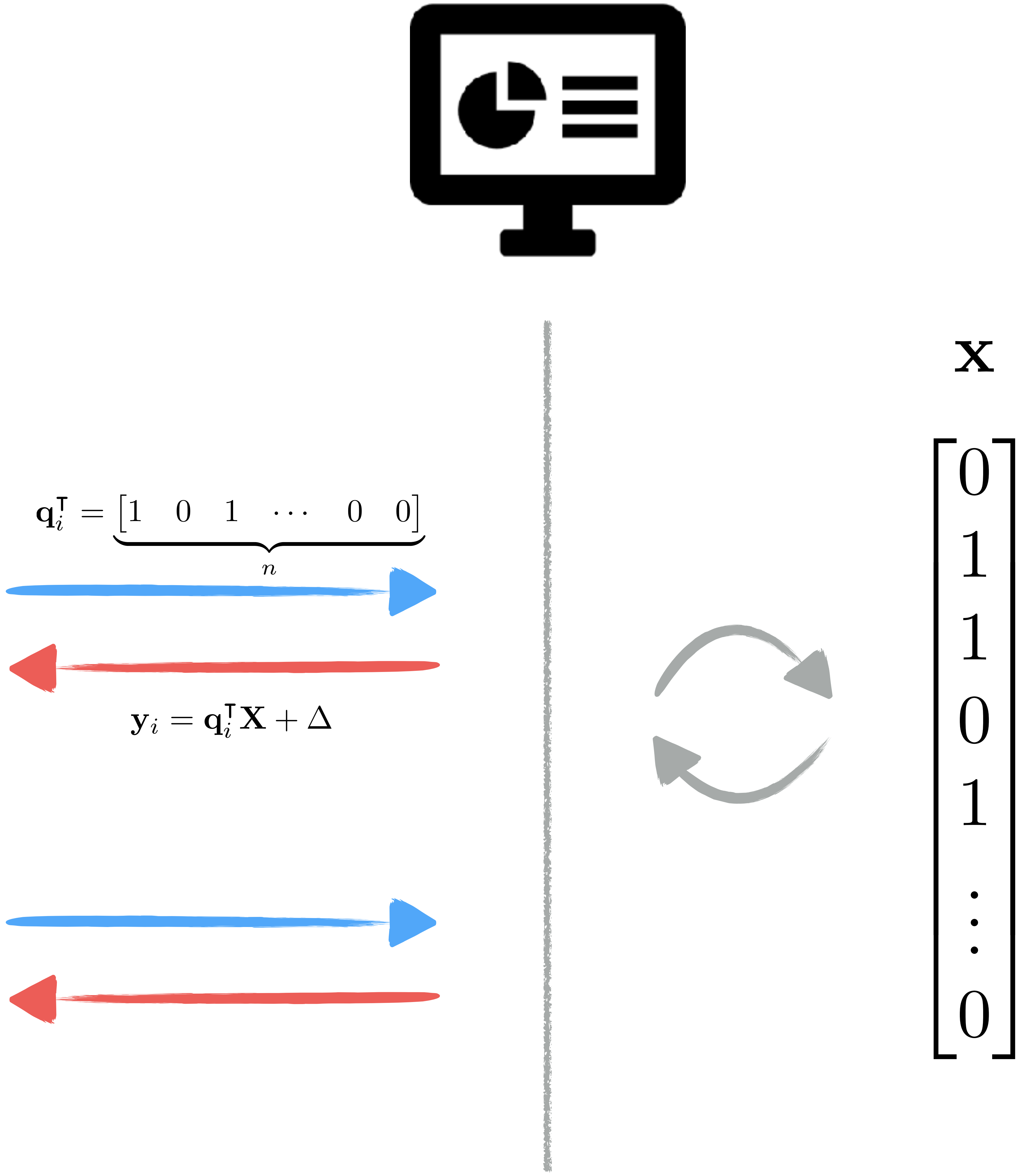
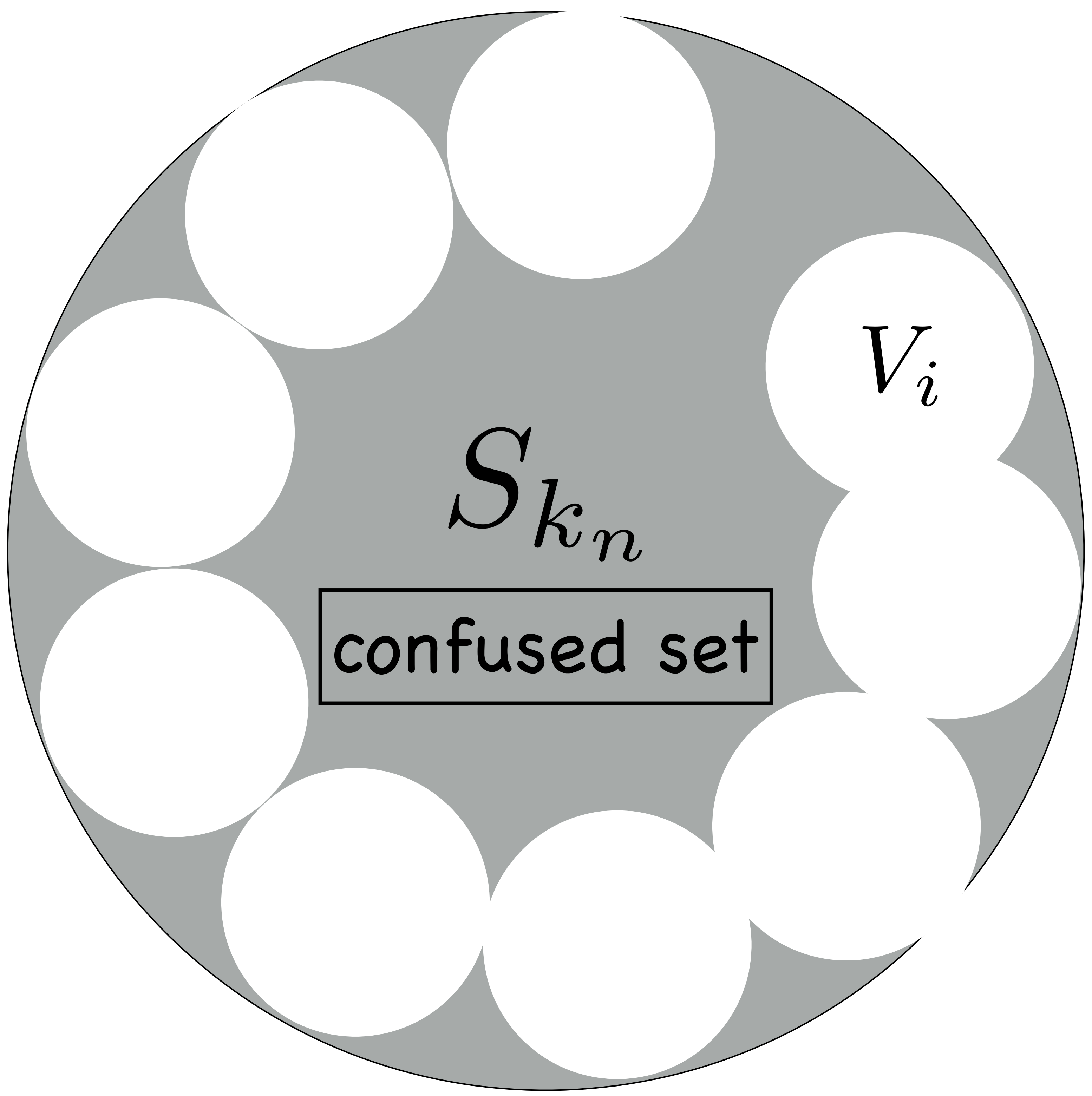
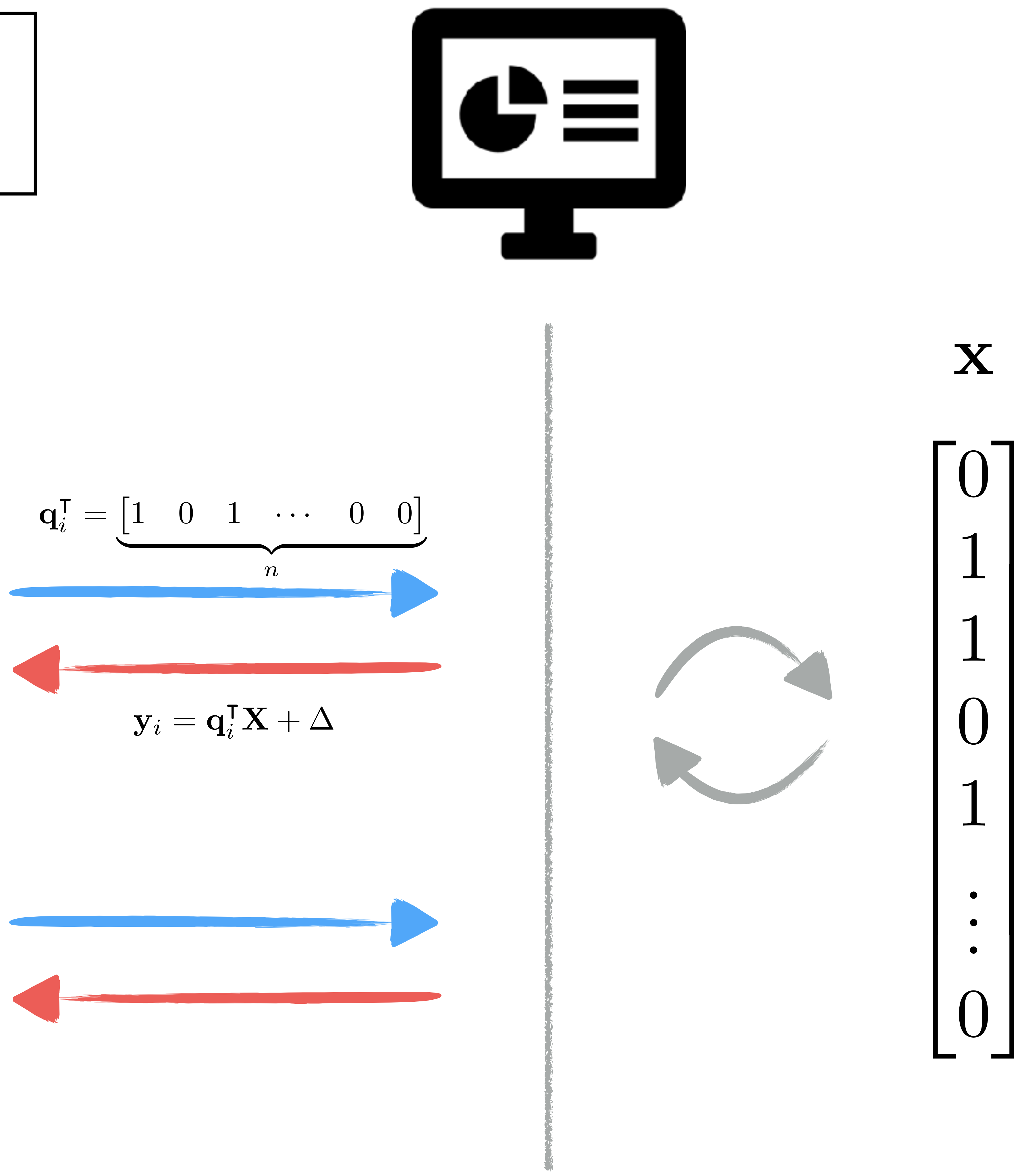




Impossibility of Polynomial Query

at least $\frac{|S_{k_n}|}{\max_i |V_i|}$ queries are required





Impossibility of Polynomial Query

- Therefore, we have the following lower bound on $T_n^*(k_n, \delta_n)$:

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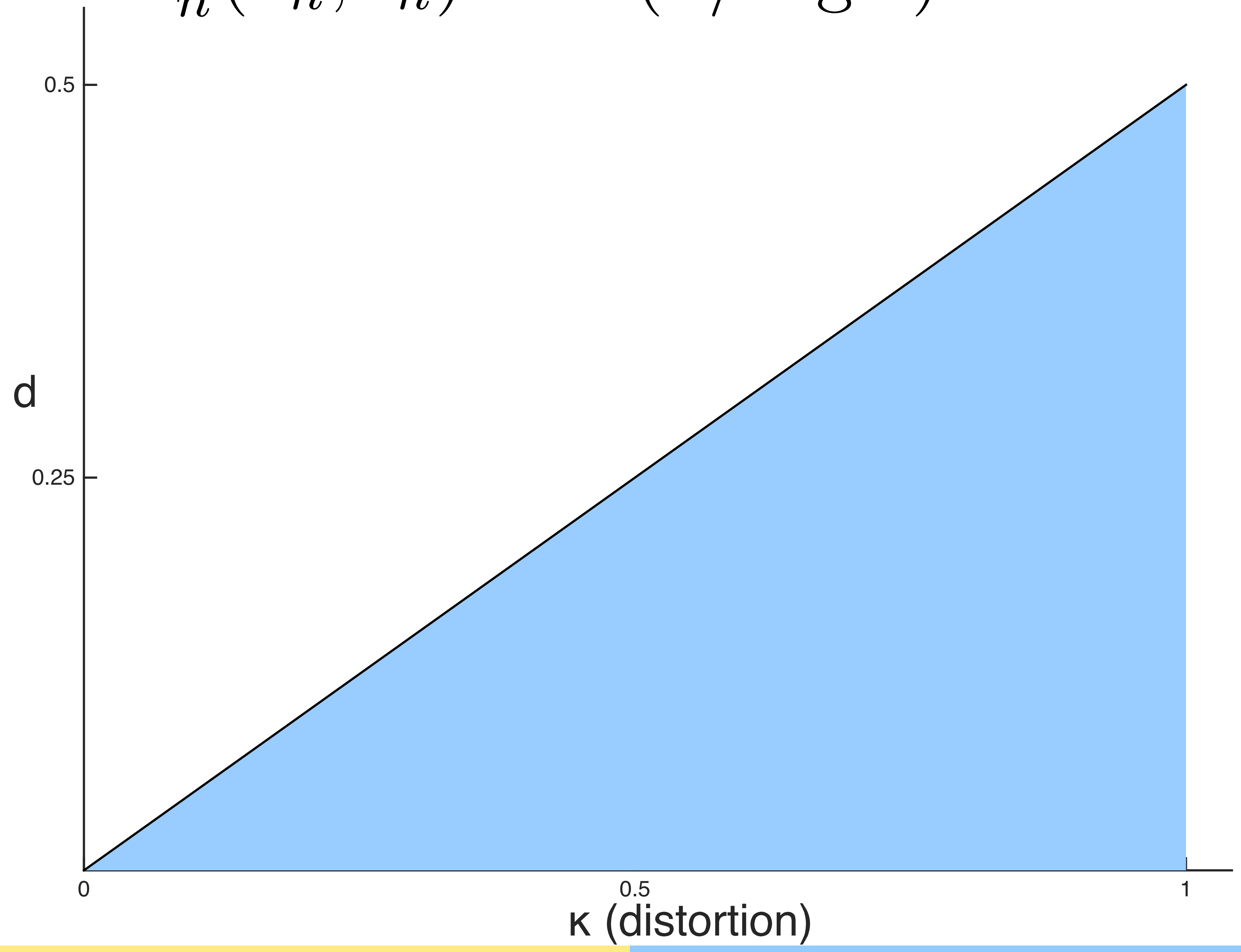
solve the optimization over V, and apply Chernoff ineq.

$$\geq C \exp\left(\frac{\delta_n^2}{k_n}\right) = C \exp\left(n^{2d-\kappa}\right)$$

Regime 2: Achievability and Converse

- Regime 2: $d < \frac{1}{2}\kappa$ (the noise is small enough)

$$T_n^*(k_n, \delta_n) = \Theta\left(n/\log n\right)$$



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  - Applying Chernoff bound on failure event

Converse Lower Bound

- Necessary condition :

$$\forall \mathbf{x}, \tilde{\mathbf{x}} \in \mathcal{X},\ \|\mathbf{x} - \tilde{\mathbf{x}}\|_1 > k_n \implies \|\mathbf{Q}\mathbf{x} - \mathbf{Q}\tilde{\mathbf{x}}\|_\infty > 2\delta_n$$

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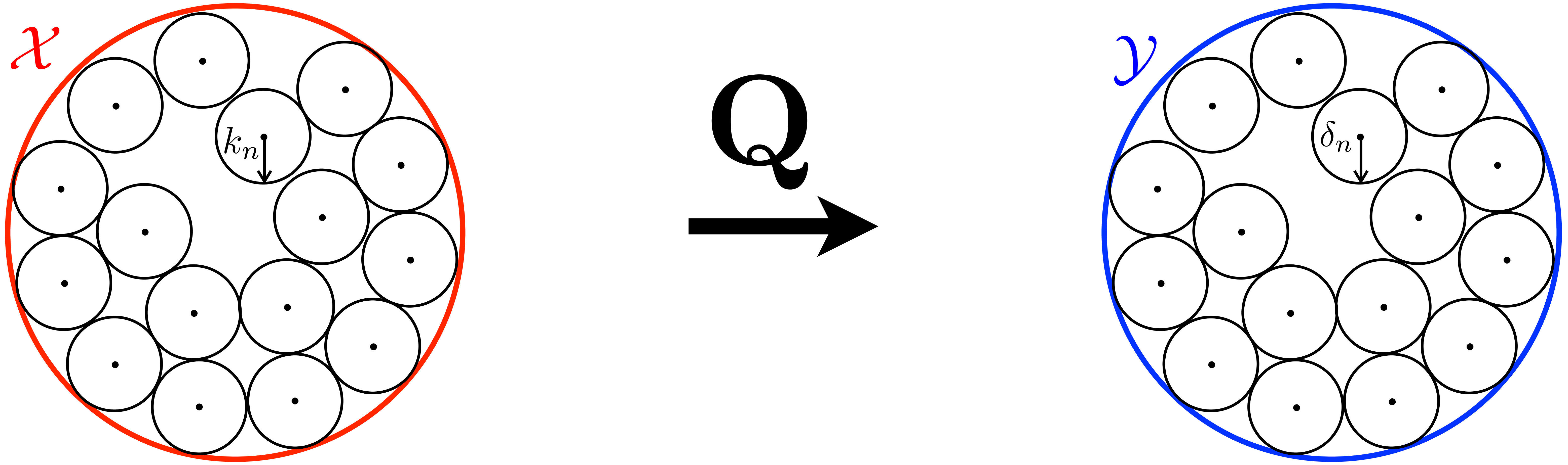
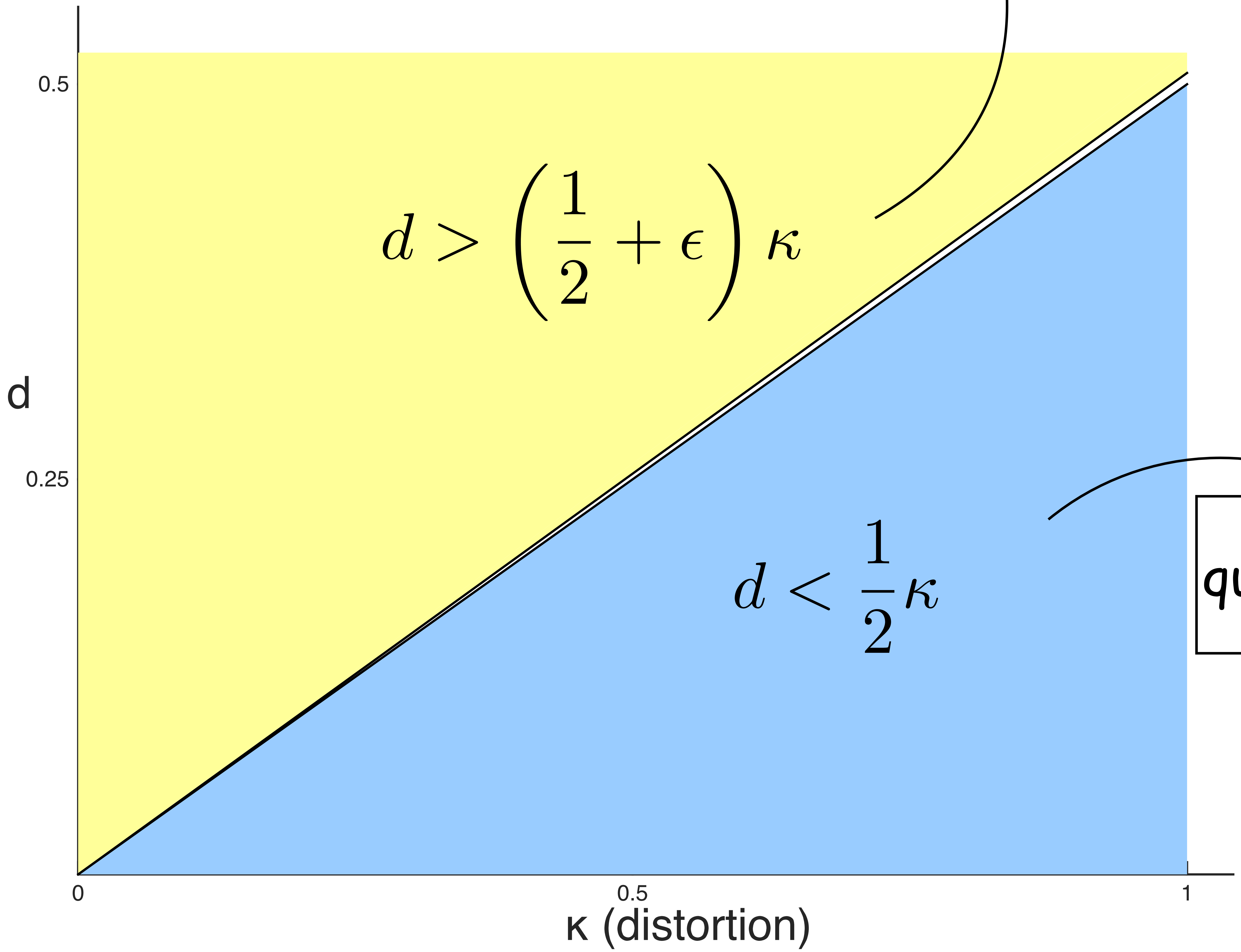


Summary

$$\delta_n = \Theta\left(n^d\right), \; k_n = \Theta\left(n^\kappa\right)$$

query complexity: non-polynomial $\Omega\left(\exp\left(n^\epsilon\right)\right)$



$$d > \left(\frac{1}{2} + \epsilon\right)\kappa$$

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Reference

[1] I.-H. Wang, S.-L. Huang et. al. “Data extraction via histogram and arithmetic mean queries: Fundamental limits and algorithms,” *Proceedings of IEEE International Symposium on Information Theory*, July 2016.

[2] Ahmed El Alaoui , et. al “Decoding from Pooled Data: Phase Transitions of Message Passing ,” *Proceedings of IEEE International Symposium on Information Theory,* June 2017

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Question ?

Thank you for your attention !